Characterization of Upper Detour Monophonic Domination Number

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ABSTRACT
This paper introduces the concept of upper detour monophonic domination number of a graph. For a connected graph \( G \) with vertex set \( V(G) \), a set \( M \subseteq V(G) \) is called minimal detour monophonic dominating set, if no proper subset of \( M \) is a detour monophonic dominating set. The maximum cardinality among all minimal monophonic dominating sets is called upper detour monophonic domination number and is denoted by \( \gamma^+_dm(G) \). For any two positive integers \( p \) and \( q \) with \( 2 \leq p \leq q \) there is a connected graph \( G \) with \( \gamma_m(G) = \gamma_{dm}(G) = p \) and \( \gamma^+_dm(G) = q \). For any three positive integers \( p, q, r \) with \( 2 < p < q < r \), there is a connected graph \( G \) with \( m(G) = p, \gamma_{dm}(G) = q \) and \( \gamma^+_dm(G) = r \). Let \( p \) and \( q \) be two positive integers with \( 2 < p < q \) such that \( \gamma_{dm}(G) = p \) and \( \gamma^+_dm(G) = q \). Then there is a minimal DMD set whose cardinality lies between \( p \) and \( q \). Let \( p, q \) and \( r \) be any three positive integers with \( 2 \leq p \leq q \leq r \). Then, there exist a connected graph \( G \) such that \( \gamma_{dm}(G) = p, \gamma^+_dm(G) = q \) and \( |V(G)| = r \).

RESUMEN
Este artículo introduce el concepto de número de dominación de desvío monofónico superior de un grafo. Para un grafo conexo \( G \) con conjunto de vértices \( V(G) \), un conjunto \( M \subseteq V(G) \) se llama conjunto dominante de desvío monofónico minimal, si ningún subconjunto propio de \( M \) es un conjunto dominante de desvío monofónico. La cardinalidad máxima entre todos los conjuntos dominantes de desvío monofónico minimales se llama número de dominación de desvío monofónico superior y se denota por \( \gamma^+_dm(G) \). Para cualquier par de enteros positivos \( p \) y \( q \) con \( 2 \leq p \leq q \) existe un grafo conexo \( G \) con \( \gamma_m(G) = \gamma_{dm}(G) = p \) y \( \gamma^+_dm(G) = q \). Para cualquiera tres enteros positivos \( p, q, r \) con \( 2 < p < q < r \), existe un grafo conexo \( G \) con \( m(G) = p, \gamma_{dm}(G) = q \) y \( \gamma^+_dm(G) = r \). Sean \( p \) y \( q \) dos enteros positivos con \( 2 < p < q \) tales que \( \gamma_{dm}(G) = p \) y

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\[ \gamma_{dm}^+(G) = q. \] Entonces existe un conjunto DMD mínimo cuya cardinalidad se encuentra entre \( p \) y \( q \). Sean \( p, q \) y \( r \) tres enteros positivos cualquiera con \( 2 \leq p \leq q \leq r \). Entonces existe un grafo conexo \( G \) tal que \( \gamma_{dm}(G) = p, \gamma_{dm}^+(G) = q \) y \( |V(G)| = r \).

**Keywords and Phrases:** Monophonic number, Domination Number, Detour monophonic number, Detour monophonic domination number, Upper detour monophonic domination number.

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## 1 Introduction

Consider an undirected connected graph \( G(V, E) \) without loops or multiple edges. Let \( P : u_1, u_2, ... u_n \) be a path of \( G \). An edge \( e \) is said to be a chord of \( P \) if it is the join of two non adjacent vertices of \( P \). A path is said to be monophonic path if there is no chord. If \( S \) is a set of vertices of \( G \) such that each vertex of \( G \) lies on an \( u-v \) monophonic path in \( G \) for some \( u, v \in S \), then \( S \) is called monophonic set. Monophonic number is the minimum cardinality among all the monophonic sets of \( G \). It is denoted by \( m(G) \) [1,2].

A vertex \( v \) in a graph \( G \) dominates itself and all its neighbours. A set \( T \) of vertices in a graph \( G \) is a dominating set if \( N[T] = V(G) \). The minimum cardinality among all the dominating sets of \( G \) is called domination number and is denoted by \( \gamma(G) \) [4]. A set \( T \subset V(G) \) is a monophonic dominating set of \( G \) if \( T \) is both monophonic set and dominating set. The monophonic domination number is the minimum cardinality among all the monophonic dominating sets of \( G \) and is denoted by \( \gamma_m(G) \) [5,6]. A monophonic set \( M \) in a connected graph \( G \) is minimal monophonic set if no proper subset of \( M \) is a monophonic set. The upper monophonic number is the maximum cardinality among all minimal monophonic sets and is denoted by \( m^+(G) \) [9].

The shortest \( x-y \) path is called geodetic path and longest \( x-y \) monophonic path is called detour monophonic path. If every vertex of \( G \) lies on a \( x-y \) detour monophonic path in \( G \) for some \( x, y \in M \subseteq V(G) \), \( M \) could be identified as a detour monophonic set. The minimum cardinality among all the detour monophonic set is the detour monophonic number and is denoted by \( dm(G) \). A minimal detour monophonic set \( D \) of a connected graph \( G \) is a subset of \( V(G) \) whose any proper subset is not a detour monophonic set of \( G \). The maximum cardinality among all minimal detour monophonic sets is called upper detour monophonic set, denoted by \( dm^+(G) \) [10].

If \( D \) is both a detour monophonic set and a dominating set, it could be a detour monophonic dominating set. The minimum cardinality among all detour monophonic dominating sets of \( G \) is the detour monophonic dominating number( DMD number) and is denoted by \( \gamma_{dm}(G) \) [7,8]. A vertex \( v \) is an extreme vertex if the sub graph induced by its neighbourhood is complete. A vertex \( u \) in a connected graph \( G \) is a cut-vertex of \( G \), if \( G - u \) is disconnected. In this article, we consider
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G as a connected graph of order \( n \geq 2 \) if otherwise not stated. For basic notations and terminology refer [3].

**Theorem 1.1** (8). Each extreme vertex of a connected graph \( G \) belongs to every detour monophonic dominating set of \( G \).

**Example 1.1.** Consider the graph \( G \) given in Figure 1. Here \( M_1 = \{v_1, v_4\} \) is a monophonic set. Therefore \( m(G) = 2 \). \( M_1 \) also dominate \( G \). Hence \( \gamma(G) = 2 \). The set \( M_2 = \{v_1, v_2, v_3\} \) is a minimum detour monophonic set. Thus \( \gamma_{dm}(G) = 3 \). \( M_2 \) does not dominate \( G \). \( M_2 \cup \{v_4\} \) is a minimum DMD set. Therefore \( \gamma_{dm}(G) = 4 \).

### 2 UDMD Number of a Graph

**Definition 2.1.** A detour monophonic dominating set \( M \) in a connected graph \( G \) is called minimal detour monophonic dominating set if no proper subset of \( M \) is a detour monophonic dominating set. The maximum cardinality among all minimal detour monophonic dominating sets is called upper detour monophonic domination number and is denoted by \( \gamma_{dm}^+(G) \).

![Graph G with UDMD number 5](image)

**Example 2.1.** Consider the graph \( G \) given in Figure 1. The set \( M = \{v_1, v_5, v_6, v_7, v_8\} \) is a minimal DMD set with maximum cardinality. Therefore \( \gamma_{dm}^+(G) = 5 \).

**Theorem 2.1.** Let \( G \) be a connected graph and \( v \) an extreme vertex of \( G \). Then \( v \) belongs to every minimal detour monophonic dominating set of \( G \).

**Proof.** Every minimal detour monophonic dominating set is a minimum detour monophonic set. Since each extreme vertex belongs to every minimum detour monophonic dominating set, the result follows. ■
Theorem 2.2. Let \( v \) be a cut-vertex of a connected graph \( G \). If \( M \) is a minimal DMD set of \( G \), then each component of \( G - v \) have an element of \( M \).

Proof. Suppose let \( A \) is a component of \( G - v \) having no vertices of \( M \). Let \( u \) be any one of the vertex in \( A \). Since \( M \) is a minimal DMD set, there exist two vertices \( p, q \) in \( M \) such that \( u \) lies on a \( p - q \) detour monophonic path \( P : p, u, u_1, ..., u, ..., u_m = q \) in \( G \). Consider two sub-paths \( P_1 : p - u \) and \( P_2 : u - q \) of \( P \). Given \( v \) is a cut-vertex of \( G \). Therefore both \( P_1 \) and \( P_2 \) contain \( v \). Hence \( P \) is not a path. This is a contradiction. That is, each component of \( G - v \) have an element of every minimal DMD set.

Theorem 2.3. For a connected graph \( G \) of order \( n \), \( \gamma_{dm}(G) = n \) if and only if \( \gamma_{dm}(G) = n \).

Proof. First, suppose \( \gamma_{dm}(G) = n \). That is \( M = V(G) \) is the unique minimal DMD set of \( G \), so that no proper subset of \( M \) is a DMD set. Hence \( M \) is the unique DMD set. Therefore \( \gamma_{dm}(G) = n \). Conversely, let \( \gamma_{dm}(G) = n \). Since every DMD set is a minimal DMD set, \( \gamma_{dm}(G) \leq \gamma_{dm}(G) \). Therefore \( \gamma_{dm}(G) \geq n \). Since \( V(G) \) is the maximum DMD set, \( \gamma_{dm}(G) = n \). □

3 UDMD Number of Some Standard Graphs

Example 3.1. Complete bipartite graph \( K_{m,n} \)

For complete bipartite graph \( G = K_{m,n} \),

\[
\gamma_{dm}^+(G) = \begin{cases} 
2, & \text{if } m = n = 1; \\
n, & \text{if } n \geq 2, m = 1; \\
4, & \text{if } m = n = 3 \\
\max\{m, n\}, & \text{if } m, n \geq 2, m, n \neq 3
\end{cases}
\]

Proof. Case (i): Let \( m = n = 1 \). Then \( K_{m,n} = K_2 \). Therefore \( \gamma_{dm}^+(G) = 2 \).

Case (ii): Let \( n \geq 2, m = 1 \). This graph is a rooted tree. There are \( n \) end vertices. All these are extreme vertices. Therefore they belong to every DMD set and consequently every minimal DMD set.

Case (iii): If \( m = n = 3 \), then exactly two vertices from both the particians form a minimal DMD set.

Case (iv): Let \( m, n \geq 2, m, n \neq 3 \). Assume that \( m \leq n \). Let \( A = \{a_1, a_2, ... a_m\} \) and \( B = \{b_1, b_2, ... b_n\} \) be the partitions of \( G \). First, prove \( M = B \) is a minimal DMD set. Take a vertex \( a_j, 1 \leq j \leq m \), which lies in a detour monophonic path \( b_1a_jb_k \) for \( k \neq j \) so that \( M \) is a detour monophonic set. They also dominate \( G \). Hence \( M \) is a DMD set.
Next, let $S$ be any minimal DMD set such that $|S| > n$. Then $S$ contains vertices from both the sets $A$ and $B$. Since $A$ and $B$ are themselves minimal DMD sets, they do not completely belong to $S$. Note that if $S$ contains exactly two vertices from $A$ and $B$, then it is a minimum DMD set. Thus $\gamma^+_d(G) = n = \max\{m, n\}$. 

**Example 3.2. Complete graph $K_n$**

For complete graph $G = K_n$, $\gamma^+_d(G) = n$.

**Proof.** For a complete graph $G$, every vertex in $G$ is an extreme vertex. By theorem 2.1 they belong to every minimal DMD set. 

**Example 3.3. Cycle graph $C_n$**

For Cycle graph $G = C_n$ with $n$ vertices,

$$\gamma^+_d(G) = \begin{cases} 3, & \text{if } n \leq 7, n \neq 4 \\ 2, & \text{if } n = 4 \\ 4 + \frac{n - 7 - r}{3}, & \text{if } n \geq 8, \; n - 7 \equiv r \mod(3) \end{cases}$$

**Proof.** For $n \leq 7$ the results are trivial. For $n \geq 8$, let $C_n : v_1, v_2, v_3, ..., v_n, v_1$ be the cycle with $n$ vertices. Then the set of vertices $\{v_1, v_3, v_{n-1}\}$ is a minimal detour monophonic set but not dominating. This set dominates only seven vertices. There are $n - 7$ remaining vertices. If $r$ is the reminder when $n - 7$ is divided by 3, then $\frac{n - 7 - r}{3} + 1$ vertices dominate the remaining vertices. Therefore every minimal DMD set contains $4 + \frac{n - 7 - r}{3}$ vertices. 

**4 Characterization of $\gamma^+_d(G)$**

**Theorem 4.1.** For any two positive integers $p$ and $q$ with $2 \leq p \leq q$ there is a connected graph $G$ with $\gamma_m(G) = \gamma_d(G) = p$ and $\gamma^+_d(G) = q$.

**Proof.** Construct a graph $G$ as follows. Let $C_6 : u_1, u_2, u_3, u_4, u_5, u_6, u_1$ be the cycle of order 6. Join $p - 1$ disjoint vertices $M_1 = \{x_1, x_2, ..., x_{p-1}\}$ with the vertex $u_1$. Let $M_2 = \{y_1, y_2, ..., y_{q-p-1}\}$ be a set of $q - p - 1$ disjoint vertices. Add each vertex in $M_2$ with $u_4$ and $u_6$. Let $x_{p-1}$ be adjacent with $u_2$ and $u_6$. This is the graph $G$ given in Figure 2.

Since all vertices except $x_{p-1}$ in $M_1$ are extreme, they belong to every minimum monophonic dominating set and DMD set. The set $M = M_1 \cup \{u_4\}$ is a minimum monophonic dominating set. Therefore $\gamma_m(G) = p$. Moreover, the set of all vertices in $M$ form a DMD set and is minimum. That is $\gamma_d(G) = p$. 

Next, we prove that $\gamma^+_\text{dm}(G) = q$. Clearly $N = M_1 \cup M_2 \cup \{u_5, u_6\}$ is a DMD set. $N$ is also a minimal DMD set of $G$. For the proof, let $N'$ be any proper subset of $N$. Then there exists at least one vertex $u \in N$ and $u \notin N'$. If $u = y_i$, for $1 \leq i \leq q - p - 1$, then $y_i$ does not lie on any $x - y$ detour monophonic path for some $x, y \in N'$. Similarly if $u \in \{u_5, u_6, x_{p-1}\}$, then that vertex does not lie on any detour monophonic path in $N'$. Thus $N$ is a minimal DMD set. Therefore $\gamma^+_\text{dm}(G) \geq q$.

$\gamma_m(G) = \gamma_\text{dm}(G) = p$ and $\gamma^+_\text{dm}(G) = q$.

Note that $N$ is a minimal DMD set with maximum cardinality. On the contrary, suppose there exists a minimal DMD set, say $T$, whose cardinality is strictly greater than $q$. Then there is a vertex $u \in T, u \notin N$. Therefore $u \in \{u_2, u_3, u_4\}$. If $u = u_4$, then $M_1 \cup \{u_4\}$ is a DMD set properly contained in $T$ which is a contradiction. If $u = u_3$, then the set $M_1 \cup \{u_3, u_5\}$ is a DMD set which is a proper subset of $T$ and is a contradiction. If $u = u_2$, then the set $(N - \{u_6\}) \cup \{u_2\}$ is a DMD set properly contained in $T$ and is a contradiction. Thus $\gamma^+_\text{dm}(G) = q$.

**Theorem 4.2.** For any three positive integers $p, q, r$ with $2 < p < q < r$, there is a connected graph $G$ with $m(G) = p$, $\gamma_\text{dm}(G) = q$ and $\gamma^+_\text{dm}(G) = r$.

**Proof.** Let $G$ be the graph constructed as follows. Take $q - p$ copies of a cycle of order 5 with each cycle $C_i$ has a vertex set $\{d_i, e_i, f_i, g_i, h_i\}$, for $1 \leq i \leq q - p$. Join each $e_i$ with all other vertices in $C_i$. Also join the vertex $f_{i-1}$ of $C_{i-1}$ with the vertex $d_i$ of $C_i$. Let $\{u, v\}$ and $\{b_1, b_2, ..., b_{r-q+1}\}$ be two sets of mutually non adjacent vertices. Join each $b_i$ with $u$ and $v$, for $1 \leq i \leq r - q + 1$. Join another $p - 2$ pendant vertices with $u$ and one pendant vertex with $d_1$. This is the graph $G$ given in Figure 3.

The set $M_1 = \{a_0, a_1, a_2, ..., a_{p-2}\}$ is the set of all extreme vertices and belongs to every monophonic dominating set and DMD set (Theorem 1.1). Clearly $M_1$ is not monophonic. But $M_1 \cup \{v\}$ is a monophonic set and is minimum. Therefore $m(G) = p$. Take $M_2 = \{e_1, e_2, ..., e_{q-p}\}$. Then $M_1 \cup M_2 \cup \{v\}$ is a DMD set and is minimum. Therefore $\gamma_\text{dm}(G) = p - 1 + q - p + 1 = q$. ■
Let $M_3 = \{b_1, b_2, ..., b_{r-q+1}\}$. Then $M = M_1 \cup M_2 \cup M_3$ is a DMD set. Now $M$ is a minimal DMD set. On the contrary, suppose $N$ is any proper DMD subset of $M$ so that there exists at least one vertex in $M$ which does not belong to $N$. Let $u \in M$ and $u \notin N$. Clearly $u \notin M_4$ since it is the set of all extreme vertices. If $u = e_i$ for some $i$, then the vertex $e_i$ does not belong to any detour monophonic path induced by $N$. Therefore $u \notin M_2$. Similarly $u \notin M_3$. This is a contradiction. Hence $M$ is a minimal DMD set with maximum cardinality. Therefore $\gamma_{dm}^+(G) = |M_1| + |M_2| + |M_3| = (p-1) + (q-p) + (r-q+1) = r$.

**Theorem 4.3.** Let $p$ and $q$ be two positive integers with $2 < p < q$ such that $\gamma_{dm}(G) = p$ and $\gamma_{dm}^+(G) = q$. Then there is a minimal DMD set whose cardinality lies between $p$ and $q$.

**Proof.** Consider three sets of mutually disjoint vertices $M_1 = \{a_1, a_2, ..., a_{q-n+1}\}$, $M_2 = \{b_1, b_2, ..., b_{n-p+1}\}$ and $M_3 = \{x, y, z\}$. Join each vertex $a_i$ with $x$ and $z$ and each vertex $b_j$ with $y$ and $z$. Add $p-2$ pendent vertices $M_4 = \{c_1, c_2, ..., c_{p-2}\}$ with the vertex $y$. This is the graph $G$ given in Figure 4. Since $M_4$ is the set of all extreme vertices, it belongs to every DMD set. But $M_4$ is not a DMD set. The set $M = M_4 \cup \{x, z\}$ is a minimum DMD set. Therefore $\gamma_{dm}(G) = p$.

Consider the set $N = M_1 \cup M_2 \cup M_4$. We claim $N$ is a minimal DMD set with maximum cardinality. On the contrary, suppose there is a set $N' \subset N$ which is a DMD set of $G$. Then there exists at least one vertex, say $u$ in $N$ which does not belong to $N'$. Clearly $u \notin M_4$ since it is the set of all extreme vertices. If $u \in M_1$, then $u = a_i$ for some $i$. Then the vertex $a_i$ does not lie on any detour monophonic path, which is a contradiction. Similarly, if $u \in M_2$, we get a contradiction. Thus $N$ is a minimal DMD set. Therefore $\gamma_{dm}^+(G) \geq q$. \]
Figure 4: Graph $G$ with $\gamma_{dm}(G) = p$ and $\gamma^+_{dm}(G) = q$

Next, we claim that $N$ has the maximum cardinality of any minimal DMD set. If $\gamma^+_{dm}(G) > q$, there is at least one vertex $v \in V(G)$, $v \notin N$ and belongs to a minimal DMD set. Therefore $v \in M_3$. If $v = x$, then the set $M_2 \cup M_4 \cup \{v\}$ is a minimal DMD set having less than $q$ vertices. Similarly if $v = z$, then the set $M_1 \cup M_4 \cup \{v\}$ is a minimal DMD set. For $v = y$, the set $N \cup \{y\}$ is not a minimal DMD set. Therefore $\gamma^+_{dm}(G) \leq q$.

Let $n$ be any number which lies between $p$ and $q$. Then there is a minimal DMD set of cardinality $n$. For the proof, consider the set $T = M_2 \cup M_4 \cup \{x\}$. $T$ is a minimal DMD set. If $T$ is not a minimal DMD set, there is a proper subset $T'$ of $T$ such that $T'$ is a minimal DMD set. Let $u \in T$ and $u \notin T'$. Since each vertex in $M_4$ is an extreme vertex, $v \notin M_4$. If $u = x$, then the vertex $u$ is not an internal vertex of any detour monophonic path in $T'$. A similar argument may be made if $u \in M_2$. This leads to a contradiction. Therefore $T$ is a minimal DMD set with cardinality $(n - p + 1) + (p - 2) + 1 = n$.

Theorem 4.4. Let $p$, $q$ and $r$ be any three positive integers with $2 \leq p \leq q \leq r$. Then, there exists a connected graph $G$ such that $\gamma_{dm}(G) = p$, $\gamma^+_{dm}(G) = q$ and $|V(G)| = r$.

Proof. Let $K_{1,p}$ is a star graph with leaves set $M_1 = \{u_1, u_2, \ldots, u_p\}$ and let $u$ be the support vertex of $K_{1,p}$. Insert $r - q - 1$ vertices $M_2 = \{v_1, v_2, \ldots, v_{r-q-1}\}$ in the edges $uu_i$ respectively for $1 \leq i \leq r - q - 1$. Add $q-p$ vertices $M_3 = \{x_1, x_2, \ldots, x_{q-p}\}$ with this graph and join each $x_i$ with $u$ and $u_1$. This is the graph $G$ as shown in Figure 5. Here $|V(G)| = (q-p) + p + (r-q-1) + 1 = r$. The length of a detour monophonic path is 4.
Let $T = M_1 - \{u_1\}$. All the vertices in $T$ are extreme vertices and belong to all DMD sets and minimal DMD sets. Clearly $M_1$ is a DMD set with minimum cardinality. Therefore $\gamma_{dm}(G) = p$. Let $N = T \cup M_3 \cup \{v_1\}$. Then $|N| = (p - 1) + (q - p) + 1 = q$. We claim that $N$ is a minimal DMD set with maximum cardinality.

On the contrary, suppose there is a proper subset $N'$ of $N$ which is a minimal DMD set of $G$. Then there exists at least one vertex $x \in N, x \notin N'$. Clearly $x \notin T$. If $x \in M_3$, then $x = x_i$ for some $i, 1 \leq i \leq q - p$. Then the vertex $x_i$ does not lie on any $u - v$ detour monophonic path for $u, v \in N'$. If $x = v_1$ then $v_1$ does not lies on any detour monophonic path in $N'$. Thus no such vertex $x$ exists. This is a contradiction. Therefore $\gamma^+_{dm}(G) \geq q$.

To prove maximum cardinality of $N$, suppose there exists a minimal DMD set $S$ with $|S| > q$. Since $S$ contains $T$, the set of all extreme vertices, the vertex $x$ lies on some $u - v$ detour monophonic path for all $x \in \{u, v_2, v_3, ..., v_{r-q-1}\}$. Now $S$ is a minimal DMD set having more than $q$ vertices and $u, v_2, v_3, ..., v_{r-q-1} \notin S$. Therefore $S = \{v_1\} \cup M_3 \cup \{u_1\} \cup T$. Then $N$ is properly contained in $S$. This is a contradiction. Therefore $\gamma^+_{dm}(G) = q$. Hence the proof. \square
References


