D-metric Spaces and Composition Operators Between Hyperbolic Weighted Family of Function Spaces

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ABSTRACT

The aim of this paper is to introduce new hyperbolic classes of functions, which will be called $B_{\alpha, \log}^*$ and $F_{\log}^*(p, q, s)$ classes. Furthermore, we introduce D-metrics space in the hyperbolic type classes $B_{\alpha, \log}^*$ and $F_{\log}^*(p, q, s)$. These classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, necessary and sufficient conditions are given for the composition operator $C_\phi$ to be bounded and compact from $B_{\alpha, \log}^*$ to $F_{\log}^*(p, q, s)$ spaces.

RESUMEN

El objetivo de este artículo es introducir nuevas clases hiperbólicas de funciones, que serán llamadas clases $B_{\alpha, \log}^*$ y $F_{\log}^*(p, q, s)$. A continuación, introducimos D-espacios métricos en las clases de tipo hiperbólicas $B_{\alpha, \log}^*$ y $F_{\log}^*(p, q, s)$. Mostramos que estas clases son espacios métricos completos con respecto a las métricas correspondientes. Más aún, damos condiciones necesarias y suficientes para que el operador composición $C_\phi$ sea acotado y compacto desde el espacio $B_{\alpha, \log}^*$ a $F_{\log}^*(p, q, s)$.

Keywords and Phrases: D-metric spaces, Logarithmic hyperbolic classes, Composition operators.

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1 Introduction

Let $\phi$ be an analytic self-map of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $\partial \mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$ and let $B(\mathbb{D})$ be the subset of $H(\mathbb{D})$ consisting of those $f \in H(\mathbb{D})$ for which $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Let the Green’s function of $\mathbb{D}$ be defined as $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{\bar{a}-z}$ is the Möbius transformation related to the point $a \in \mathbb{D}$.

A linear composition operator $C_\phi$ is defined by $C_\phi(f) = (f \circ \phi)$ for $f$ in the set $H(\mathbb{D})$ of analytic functions on $\mathbb{D}$ (see [9]). A function $f \in B(\mathbb{D})$ belongs to $\alpha$-Bloch space $B_\alpha$, $0 < \alpha < \infty$, if

$$||f||_{B_\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f'(z)| < \infty.$$  

The little $\alpha$-Bloch space $B_{\alpha,0}$ consisting of all $f \in B_\alpha$ so that

$$\lim_{|z| \to 1^-} (1 - |z|^2) |f'(z)| = 0.$$  

**Definition 1.** [15] For an analytic function $f$ on $\mathbb{D}$ and $0 < \alpha < \infty$, if

$$||f||_{B_{\log}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| \left( \log \frac{2}{1 - |z|^2} \right) < \infty,$$

then, $f$ belongs to the weighted $\alpha$-Bloch spaces $B_{\log}^\alpha$.

If $\alpha = 1$, the weighted Bloch space $B_{\log}$ is the set for all analytic functions $f$ in $\mathbb{D}$ for which $||f||_{B_{\log}} < \infty$.

The expression $||f||_{B_{\log}}$ defines a seminorm while the norm is defined by

$$||f||_{B_{\log}} = |f(0)| + ||f||_{B_{\log}}.$$

**Definition 2.** [14] For $0 < p, q < \infty$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is in $F(p, q, s)$, if

$$\sup_{a \in \mathbb{D}} \int_\mathbb{D} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty.$$  

Moreover, if

$$\lim_{|a| \to 1^-} \int_\mathbb{D} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0,$$

then $f \in F_0(p, q, s)$.

El-Sayed and Bakhit [5] gave the following definition:
Definition 3. For $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(D)$ is said to belong to $F_{\log}(p, q, s)$, if

$$
\sup_{I \subset \partial D} \left( \log \frac{2}{|I|} \right)^p \int_{S(I)} |f'(z)|^p (1 - |z|^2)^q \left( \log \frac{1}{|z|} \right)^s dA(z) < \infty.
$$

Where $|I|$ denotes the arc length of $I \subset \partial D$ and $S(I)$ is the Carleson box defined by (see \[8, 6\])

$$
S(I) = \{ z \in D : 1 - |I| < |z| < 1, \frac{z}{|z|} \in |I| \}.
$$

The interest in the $F_{\log}(p, q, s)$-spaces rises from the fact that they cover some well known function spaces. It is immediate that $F_{\log}(2, 0, 1) = BMOA_{\log}$ and $F_{\log}(2, 0, p) = Q_p^{\log}$, where $0 < p < \infty$.

2 Preliminaries

Definition 4. [11] The hyperbolic Bloch space $B^*_\alpha$ is defined as

$$
B^*_\alpha = \{ f : f \in B(D) \text{ and } \sup_{z \in D} (1 - |z|^2)^\alpha f^*(z) < \infty \}.
$$

Denoting $f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}$, the hyperbolic derivative of $f \in B(D)$. [7]

The little hyperbolic Bloch space $B^*_{\alpha, 0}$ is a subspace of $B^*_\alpha$ consisting of all $f \in B^*_\alpha$ so that

$$
\lim_{|z| \to 1^-} (1 - |z|^2)^\alpha f^*(z) = 0.
$$

The space $B^*_\alpha$ is Banach space with the norm defined as

$$
||f||_{B^*_\alpha} = |f(0)| + \sup_{z \in D} (1 - |z|)^\alpha |f^*(z)|.
$$

Definition 5. For $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha = \frac{q+2}{p}$ and $q + s > -1$, a function $f \in H(D)$ is said to belong to $F^*(p, q, s)$, if

$$
\sup_{a \in \overline{D}} \int_{\overline{D}} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} g^*(z, a) dA(z) < \infty.
$$

Definition 6. For $f \in B(D)$ and $0 < \alpha < \infty$, if

$$
||f||_{B^*_{\alpha, \log}} = \sup_{z \in D} (1 - |z|^2)^\alpha (f^*(z))^2 \left( \log \frac{2}{1 - |z|^2} \right) < \infty,
$$

then $f$ belongs to the $B^*_{\alpha, \log}$. 
We must consider the following lemmas in our study:

**Lemma 2.1.** Let \( 0 < r \leq t \leq 1 \), then
\[
\log \frac{1}{t} \leq \frac{1}{r}(1 - t^2)
\]

**Lemma 2.2.** Let \( 0 \leq k_1 < \infty \), \( 0 \leq k_2 < \infty \), and \( k_1 - k_2 > -1 \), then
\[
C(k_1, k_2) = \int_{\mathbb{D}} \left( \log \frac{1}{|z|} \right)^{k_1} (1 - |z|^2)^{-k_2} dA(z) < \infty.
\]

To study composition operators on \( B^*_\alpha, \log \) and \( F^*_\log(p, q, s) \) spaces, we need to prove the following result:

**Theorem 1.** If \( 0 < p < \infty \), \( 1 < s < \infty \) and \( \alpha = \frac{q + 2}{p} \) with \( q + s > -1 \), then the following are equivalent:

(A) \( f \in B^*_\alpha, \log \).

(B) \( f \in F^*_\log(p, q, s) \).

(C) \( \sup_{a \in \mathbb{D}} \left( \log \frac{2}{1 - |a|^2} \right)^p \int_{\mathbb{D}} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi(z)|^2)^s dA(z) < \infty \),

(D) \( \sup_{a \in \mathbb{D}} \left( \log \frac{2}{1 - |a|^2} \right)^p \int_{\mathbb{D}} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} g^s(z, a) dA(z) < \infty \).

**Proof.** Let \( 0 < p < \infty \), \( -2 < q < \infty \), \( 1 < s < \infty \) and \( 0 < r < 1 \). By subharmonicity we have for an analytic function \( g \in \mathbb{D} \) that
\[
|g(0)|^p \leq \frac{1}{\pi r^2} \int_{D(0, r)} |g(w)|^p dA(w).
\]

For \( a \in \mathbb{D} \), the substitution \( z = \varphi_a(z) \) results in Jacobian change in measure given by
\[
dA(w) = |\varphi'_a(z)|^2 dA(z).
\]

For a Lebesgue integrable or a non-negative Lebesgue measurable function \( f \) on \( \mathbb{D} \), we thus have the following change of variable formula:
\[
\int_{D(0, r)} f(\varphi_a(w)) dA(w) = \int_{D(a, r)} f(z)|\varphi'_a(z)|^2 dA(z).
\]

Let \( g = \frac{f^* \circ \varphi_a}{1 - |f \circ \varphi_a|^2} \) then we have
\[
\left( \frac{|f'(a)|}{1 - |f(a)|^2} \right)^p = (f^*(a))^p \leq \frac{1}{\pi r^2} \int_{D(0, r)} \left( \frac{|f'(\varphi_a(w))|}{1 - |f(\varphi_a(w))|^2} \right)^p dA(w)
\]
\[
= \frac{1}{\pi r^2} \int_{D(a, r)} (f^*(z))^p |\varphi'_a(z)|^2 dA(z).
\]
Since
\[ |\varphi'_a(z)| = \frac{1 - |\varphi_a(z)|^2}{1 - |z|^2}, \]
and
\[ \frac{1 - |\varphi_a(z)|^2}{1 - |z|^2} \leq \frac{4}{1 - |a|^2}, \quad a, z \in D. \]
So we obtain that
\[ (f^*(a))^p \leq \frac{16}{\pi r^2 (1 - |a|^2)^2} \int_{D(a,r)} (f^*(z))^p dA(z). \]
Again \( f \in B^s_{\alpha, \log} \), and \( (1 - |z|^2)^2 \approx (1 - |a|^2)^2 \approx D(a,r) \), for \( z \in D(a,r) \). Thus, we have
\[
\left( \log \frac{2}{1 - |a|} \right)^p (f^*(a))^p (1 - |a|^2)^{\alpha p}
\]
\[
\leq \frac{16}{\pi r^2 (1 - |a|^2)^2} \times \left( \log \frac{2}{1 - |a|} \right)^p \int_{D(a,r)} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} dA(z)
\]
\[
\leq \frac{16}{\pi r^2} \times \left( \log \frac{2}{1 - |a|} \right)^p \int_{D(a,r)} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} \times \left( \frac{1 - |\varphi_a(z)|^2}{1 - |a|^2} \right)^s dA(z)
\]
\[
\leq M(r) \times \left( \log \frac{2}{1 - |a|} \right)^p \int_{D(a,r)} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi'_a(z)|^2)^s dA(z).
\]

Where \( M(r) \) is a constant depending on \( r \). Thus, the quantity (A) is less than or equal to constant times the quantity (C).

From the fact
\[ (1 - |\varphi_a(z)|^2) \leq 2 \log \frac{1}{|\varphi_a(z)|} = 2g(z,a) \quad \text{for} \ a, z \in D,
\]
we have
\[
\left( \log \frac{2}{1 - |a|} \right)^p \int_{D(a,r)} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z)
\]
\[
\leq \left( \log \frac{2}{1 - |a|} \right)^p \int_{D(a,r)} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} g^s(z,a) dA(z).
\]
Hence, the quantity (C) is less than or equal to a constant times (D). By taking \( \alpha = \frac{2 + \xi}{p} \), it follows \( f \in F^s_{\log}(p, q, s) \). Thus, the quantity (C) is less than or equal to a constant times the quantity (B).
Finally, from the following inequality, let \( z = \varphi_{\alpha}(w) \) then \( w = \varphi_{\alpha}(z) \). Hence,

\[
\left( \log \frac{2}{1 - |a|^2} \right)^{p} \int_{\mathbb{D}} \left( f^*(\varphi_{\alpha}(w)) \right)^{p} (1 - |\varphi_{\alpha}(w)|^2)^{\alpha p - 2} \left( \log \frac{1}{|w|} \right)^{s} |\varphi_{\alpha}(w)|^2 dA(w)
\]

\[
= \left( \log \frac{2}{1 - |a|^2} \right)^{p} \int_{\mathbb{D}} \left( f^*(\varphi_{\alpha}(w)) \right)^{p} (1 - |\varphi_{\alpha}(w)|^2)^{\alpha p} \left( \log \frac{1}{|w|} \right)^{s} |\varphi_{\alpha}(w)|^2 dA(w)
\]

\[
= \left( \log \frac{2}{1 - |a|^2} \right)^{p} \int_{\mathbb{D}} \left( f^*(\varphi_{\alpha}(w)) \right)^{p} (1 - |\varphi_{\alpha}(w)|^2)^{\alpha p} \left( \log \frac{1}{|w|} \right)^{s} \frac{1}{(1 - |\varphi_{\alpha}(w)|^2)^2} dA(w)
\]

\[
\leq ||f||_{B^*_{\alpha, log}} \left( \log \frac{2}{1 - |a|^2} \right)^{p} \int_{\mathbb{D}} \left( \log \frac{1}{|w|} \right)^{s} (1 - |w|^2)^{-2} dA(w)
\]

\[
= C(s, 2)||f||_{B^*_{\alpha, log}}^p.
\]

By lemma 2.2 \( C(s, 2) = \int_{\mathbb{D}} \left( \log \frac{1}{|w|} \right)^{s} (1 - |w|^2)^{-2} dA(w) < \infty, \quad \text{for}\ 1 < s < \infty. \)

Thus, the quantity (D) is less than or equal to a constant times the quantity (A). Hence, it is proved.

Let us give the following equivalent definition for \( F^*_{\log}(p, q, s) \).

**Definition 7.** For \( 0 < p, s < \infty, \ -2 < q < \infty, \ \alpha = \frac{q + 2}{p} \) and \( q + s > -1 \), a function \( f \in H(\mathbb{D}) \) is said to belong to \( F^*_{\log}(p, q, s) \), if

\[
\sup_{a \in \mathbb{D}} \left( \log \frac{2}{1 - |a|^2} \right)^{p} \int_{\mathbb{D}} \left( f^*(z) \right)^{p} (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_{\alpha}(z)|^2)^{s} dA(z) < \infty.
\]

**Definition 8.** A composition operator \( C_{\phi} : B^*_{\alpha, log} \rightarrow F^*_{\log}(p, q, s) \) is said to be bounded if there is a positive constant \( C \) so that \( ||C_{\phi}f||_{F^*_{\log}(p, q, s)} \leq C||f||_{B^*_{\alpha, log}} \) for all \( f \in B^*_{\alpha, log} \).

**Definition 9.** A composition operator \( C_{\phi} : B^*_{\alpha, log} \rightarrow F^*_{\log}(p, q, s) \) is said to be compact if it maps any ball in \( B^*_{\alpha, log} \) onto a precompact set in \( F^*_{\log}(p, q, s) \).

The following lemma follows by standard arguments similar to those outline in [13]. Hence, we omit the proof.

**Lemma 2.3.** Assume \( \phi \) is a holomorphic mapping from \( \mathbb{D} \) into itself. Let \( 0 < p, s, \alpha < \infty, \ -2 < q < \infty \), then \( C_{\phi} : B^*_{\alpha, log} \rightarrow F^*_{\log}(p, q, s) \) is compact if and only if for any bounded sequence \( \{f_{n}\}_{n \in \mathbb{N}} \in B^*_{\alpha, log} \) which converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \rightarrow \infty \) we have \( \lim_{n \rightarrow \infty} ||C_{\phi}f_{n}||_{F^*_{\log}(p, q, s)} = 0. \)

### 3 D-metric space

Topological properties of generalized metric space called D-metric space was introduced in [1], see for example, ([2] and [3]). This structure of D-metric space is quite different from a 2-metric space and natural generalization of an ordinary metric space in some sense.
Definition 10.\cite{4} Let $X$ denote a nonempty set and $\mathbb{R}$ the set of real numbers. A function $D : X \times X \times X \to \mathbb{R}$ is said to be a $D$-metric on $X$ if it satisfies the following properties:

(i) $D(x, y, z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x = y = z$ (nonnegativity),

(ii) $D(x, y, z) = D(x, z, y) = \cdots$ (symmetry),

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality).

A nonempty set $X$ together with a $D$-metric $D$ is called a $D$-metric space and is represented by $(X, D)$.

The generalization of a $D$-metric space with $D$-metric as a function of $n$ variables is provided in Dhage \cite{2}.

Example 1.1: \cite{4} Let $(X, d)$ be an ordinary metric space and define a function $D_1$ on $X^3$ by

$$D_1(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

for all $x, y, z \in X$. Then, the function $D_1$ is a $D$-metric on $X$ and $(X, D_1)$ is a $D$-metric space.

Example 1.2: \cite{4} Let $(X, d)$ be an ordinary metric space and define a function $D_2$ on $X^3$ by

$$D_2(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for $x, y, z \in X$. Then, $D_2$ is a metric on $X$ and $(X, D_2)$ is a $D$-metric space.

Remark 1. Geometrically, the $D$-metric $D_1$ represents the diameter of a set consisting of three points $x, y$ and $z$ in $X$ and the $D$-metric $D_2(x, y, z)$ represents the perimeter of a triangle formed by three points $x, y, z$ in $X$ as its vertices.

Definition 11. (Cauchy sequence, completeness)\cite{10} For every $m, n > N$. A sequence $(x_n)$ in a metric space $X = (X, d)$ is said to be-Cauchy if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that

$$d(x_m, x_n) < \varepsilon.$$  

The space $X$ is said to be complete if every Cauchy sequence in $X$ converges (that is, has a limit which is an element of $X$).

The following theorem can be found in \cite{4}:

Theorem 2. \cite{4} Let $d$ be an ordinary metric on $X$ and let $D_1$ and $D_2$ be corresponding associated $D$-metrics on $X$. Then, $(X, D_1)$ and $(X, D_2)$ are complete if and only if $(X, d)$ is complete.
4 D-metrics in $B^*_\alpha, \log$ and $F^*_\log(p, q, s)$

In this section, we introduce a D-metric on $B^*_\alpha, \log$ and $F^*_\log(p, q, s)$.

Let $0 < p, s < \infty$, $-2 < q < \infty$, and $0 < \alpha < 1$. First, we can find a D-metric in $B^*_\alpha, \log$, for $f, g, h \in B^*_\alpha, \log$ by defining

$$D(f, g, h; B^*_\alpha, \log) := D_{B^*_\alpha, \log}(f, g, h) + ||f - g||_{B^*_\alpha, \log} + ||g - h||_{B^*_\alpha, \log} + ||h - f||_{B^*_\alpha, \log}$$

$$+ |f(0) - g(0)| + |g(0) - h(0)| + |h(0) - f(0)|,$$

where

$$D_{B^*_\alpha, \log}(f, g, h) := d_{B^*_\alpha, \log}(f, g) + d_{B^*_\alpha, \log}(g, h) + d_{B^*_\alpha, \log}(h, f)$$

and

$$d_{B^*_\alpha, \log}(f, g, h) := \left(\sup_{z \in D} |f^*(z) - g^*(z)| + \sup_{z \in D} |g^*(z) - h^*(z)| + \sup_{z \in D} |h^*(z) - f^*(z)| \right)$$

$$\times \left(1 - |z|^2\right)^\alpha \left(\log \frac{2}{1 - |z|^2}\right).$$

Also, for $f, g, h \in F^*_\log(p, q, s)$ we introduce a D-metric on $F^*_\log(p, q, s)$ by defining

$$D(f, g, h; F^*_\log(p, q, s)) := D_{F^*_\log(p, q, s)}(f, g, h) + ||f - g||_{F^*_\log(p, q, s)} + ||g - h||_{F^*_\log(p, q, s)} +$$

$$||h - f||_{F^*_\log(p, q, s)} + |f(0) - g(0)| + |g(0) - h(0)| + |h(0) - f(0)|,$$

where

$$D_{F^*_\log(p, q, s)}(f, g, h) := d_{F^*_\log(p, q, s)}(f, g) + d_{F^*_\log(p, q, s)}(g, h) + d_{F^*_\log(p, q, s)}(h, f)$$

and

$$d_{F^*_\log(p, q, s)}(f, g) := \left(\sup_{z \in D} \varphi^p(a) \int_D |f^*(z) - g^*(z)|^p(1 - |z|^2)^q(1 - |\varphi(z)|^2)^s dA(z)\right)^\frac{1}{p}.$$ 

Proposition 1. The class $B^*_\alpha, \log$ equipped with the D-metric $D(\cdot, \cdot; B^*_\alpha, \log)$ is a complete metric space. Moreover, $B^*_\alpha, \log, 0$ is a closed (and therefore complete) subspace of $B^*_\alpha, \log$.

Proof. Let $f, g, h, a \in B^*_\alpha, \log$. Then, clearly

(i) $D(f, g, h; B^*_\alpha, \log) \geq 0$, for all $f, g, h \in B^*_\alpha, \log$. 


(ii) \( D(f, g; h; B_{\alpha, \log}^*) = D(f, h; g; B_{\alpha, \log}^*) = D(g, h; f; B_{\alpha, \log}^*) \).

(iii) \( D(f, g; h; B_{\alpha, \log}^*) \leq D(f, g; a; B_{\alpha, \log}^*) + D(f, a; h; B_{\alpha, \log}^*) + D(a, g; h; B_{\alpha, \log}^*) \)

for all \( f, g, h, a \in B_{\alpha, \log}^* \).

(iv) \( D(f, g; h; B_{\alpha, \log}^*) = 0 \) implies \( f = g = h \).

Hence, \( D \) is a \( D \)-metric on \( B_{\alpha, \log}^* \), and \( (B_{\alpha, \log}^*, D) \) is \( D \)-metric space.

To prove the completeness, we use Theorem[2] let \((f_n)_{n=1}^\infty\) be a Cauchy sequence in the metric space \((B_{\alpha, \log}^*, d)\), that is, for any \( \varepsilon > 0 \) there is an \( N = N(\varepsilon) \in \mathbb{N} \) such that \( d(f_n, f_m; B_{\alpha, \log}^*) < \varepsilon \), for all \( n, m > N \). Since \((f_n) \subset B(\mathbb{D})\), the family \((f_n)\) is uniformly bounded and hence normal in \( \mathbb{D} \). Therefore, there exists \( f \in B(\mathbb{D}) \) and a subsequence \((f_{n_j})_{j=1}^\infty\) such that \( f_{n_j} \) converges to \( f \) uniformly on compact subsets of \( \mathbb{D} \). It follows that \( f_n \) also converges to \( f \) uniformly on compact subsets, and by the Cauchy formula, the same also holds for the derivatives. Now let \( m > N \).

Then, the uniform convergence yields

\[
\left| f^*(z) - f_m^*(z) \right| (1 - |z|^2)^\alpha \left( \log \frac{2}{1 - |z|^2} \right) = \lim_{n \to \infty} \left| f^*_n(z) - f_m^*(z) \right| (1 - |z|^2)^\alpha \left( \log \frac{2}{1 - |z|^2} \right) \leq \lim_{n \to \infty} d(f_n, f_m; B_{\alpha, \log}^*) \leq \varepsilon
\]

for all \( z \in \mathbb{D} \), and it follows that \( ||f||_{B_{\alpha, \log}^*} \leq ||f_m||_{B_{\alpha, \log}^*} + \varepsilon \). Thus \( f \in B_{\alpha, \log}^* \) as desired. Moreover, the above inequality and the compactness of the usual \( B_{\alpha, \log}^* \) space imply that \((f_n)_{n=1}^\infty\) converges to \( f \) with respect to the metric \( d \), and \((B_{\alpha, \log}^*, D)\) is complete \( D \)-metric space.

Since \( \lim_{n \to \infty} d(f_n, f_m; B_{\alpha, \log}^*) \leq \varepsilon \), the second part of the assertion follows.

Next we give characterization of the complete \( D \)-metric space \( D(\cdot; F_{\log}^*(p, q, s)) \).

**Proposition 2.** The class \( F_{\log}^*(p, q, s) \) equipped with the \( D \)-metric \( D(\cdot; F_{\log}^*(p, q, s)) \) is a complete metric space. Moreover, \( F_{\log, 0}^*(p, q, s) \) is a closed (and therefore complete) subspace of \( F_{\log}^*(p, q, s) \).

**Proof.** Let \( f, g, h, a \in F_{\log}^*(p, q, s) \). Then clearly

(i) \( D(f, g; h; F_{\log}^*(p, q, s)) \geq 0 \), for all \( f, g, h \in F_{\log}^*(p, q, s) \).

(ii) \( D(f, g; h; F_{\log}^*(p, q, s)) = D(f, h; g; F_{\log}^*(p, q, s)) = D(g, h; f; F_{\log}^*(p, q, s)) \).
(iii) \( D(f, g, h; F^*_\log(p, q, s)) \leq D(f, g, a; F^*_\log(p, q, s)) + D(f, a, h; F^*_\log(p, q, s)) + D(a, g, h; F^*_\log(p, q, s)) \)

for all \( f, g, h, a \in F^*_\log(p, q, s) \).

(iv) \( D(f, g, h; F^*_\log(p, q, s)) = 0 \) implies \( f = g = h \).

Hence, \( D \) is a \( D \)-metric on \( F^*_\log(p, q, s) \), and \( (F^*_\log(p, q, s), D) \) is \( D \)-metric space.

For the complete proof, by using Theorem 2 let \( (f_n)_{n=1}^\infty \) be a Cauchy sequence in the metric space \( (F^*_\log(p, q, s), d) \), that is, for any \( \varepsilon > 0 \) there is an \( N = N(\varepsilon) \in \mathbb{N} \) so that \( d(f_n, f_m; F^*_\log(p, q, s)) < \varepsilon \), for all \( n, m > N \). Since \( (f_n) \subset B(\mathbb{D}) \), such that \( f_{n_j} \) converges to \( f \) uniformly on compact subsets of \( \mathbb{D} \). It follows that \( f_n \) also converges to \( f \) uniformly on compact subsets, now let \( m > N \), and \( 0 < r < 1 \). Then, the Fatou’s yields

\[
\begin{align*}
\int_{\mathbb{D}(0, r)} \left| f^*(z) - f_m^*(z) \right|^p \left( 1 - |z|^2 \right)^q (1 - |\varphi_a(z)|^2)^s dA(z) \\
= \int_{\mathbb{D}(0, r)} \lim_{n \to \infty} \left| f_n^*(z) - f_m^*(z) \right|^p \left( 1 - |z|^2 \right)^q (1 - |\varphi_a(z)|^2)^s dA(z) \\
\leq \lim_{n \to \infty} \int_{\mathbb{D}(0, r)} \left| f^*(z) - f_m^*(z) \right|^p \left( 1 - |z|^2 \right)^q (1 - |\varphi_a(z)|^2)^s dA(z) \leq \varepsilon^p,
\end{align*}
\]

and by taking \( r \to 1^- \), it follows that

\[
\begin{align*}
\int_{\mathbb{D}} (f^*(z))^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z) \\
\leq 2^p \varepsilon^p + 2^p \int_{\mathbb{D}} (f_m^*(z))^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z).
\end{align*}
\]

This yields

\[
||f||_{F^*_\log(p, q, s)}^p \leq 2^p ||f_m||_{F^*_\log(p, q, s)}^p + 2^p \varepsilon^p.
\]

And thus \( f \in F^*_\log(p, q, s) \). We also find that \( f_n \to f \) with respect to the metric of \( (F^*_\log(p, q, s), D) \) and \( (F^*_\log(p, q, s), D) \) is complete \( D \)-metric space. The second part of the assertion follows.

## 5 Composition operators of \( C_\phi \colon B^*_\alpha, \log \to F^*_\log(p, q, s) \)

In this section, we study boundedness and compactness of composition operators on \( B^*_\alpha, \log \) and \( F^*_\log(p, q, s) \) spaces. We need the following notation:

\[
\Phi_\phi(\alpha, p, s; a) = \ell^p(a) \int_{\mathbb{D}} |\phi'(z)|^p \frac{(1 - |z|^2)^{s-2}(1 - |\varphi_a(z)|^2)^s}{(1 - |\phi(z)|^2)^{s \alpha} \log \left( \frac{2}{1 - |\phi(z)|^2} \right)} dA(z),
\]
where \( \ell^p(a) = \left( \log \frac{2}{1-|a|^2} \right)^p \).

For \( 0 < \alpha < 1 \), we suppose there exist two functions \( f, g \in B^*_{\alpha, \log} \) such that for some constant \( C \),

\[
(|f^*(z)| + |g^*(z)|) \geq \frac{C}{(1-|z|^2)^\alpha \left( \log \frac{2}{1-|a|^2} \right)^p} > 0, \quad \text{for each } z \in \mathbb{D}.
\]

Now, we provide the following theorem:

**Theorem 3.** Assume \( \phi \) is a holomorphic mapping from \( \mathbb{D} \) into itself and let \( 0 < p, 1 < s < \infty, 0 < \alpha \leq 1 \). Then the induced composition operator \( C_\phi \) maps \( B^*_{\alpha, \log} \) into \( F^*_{\log}(p, \alpha p - 2, s) \) is bounded if and only if,

\[
\sup_{z \in \mathbb{D}} \Phi_\phi(\alpha, p, s; a) < \infty. \tag{5.1}
\]

**Proof.** First assume that \( \sup_{z \in \mathbb{D}} \Phi_\phi(\alpha, p, s; a) < \infty \) is held, and \( f \in B^*_{\alpha, \log} \) with \( ||f||_{B^*_{\alpha, \log}} \leq 1 \), we can see that

\[
||C_\phi f||_{F^*_{\log}(p, \alpha p - 2, s)}^p = \sup_{a \in \mathbb{D}} \ell^p(a) \int_{\mathbb{D}} \left| (f \circ \phi)^*(z) \right|^p (1-|z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z)
\]

\[
\leq ||f||_{B^*_{\alpha, \log}}^p \sup_{a \in \mathbb{D}} \ell^p(a) \int_{\mathbb{D}} \frac{|\phi'(z)|^p (1-|z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z)}{(1 - |\phi(z)|^2)^{p\alpha \left( \log \frac{2}{1-|a|^2} \right)^p}}
\]

\[
= ||f||_{B^*_{\alpha, \log}}^p \Phi_\phi(\alpha, p, s; a) < \infty.
\]

For the other direction, we use the fact that for each function \( f \in B^*_{\alpha, \log} \), the analytic function
Theorem 4. Let $0 < p < 1 < s < \infty, \alpha < \infty$. If $\phi$ is an analytic self-map of the unit disk, then the induced composition operator $C_\phi : B^*_\alpha, \log \rightarrow F^*_p(\log, p, \alpha p - 2, s)$ is compact if and only if $\phi \in F^*_p(\log, p, \alpha p - 2, s)$, and

$$\lim_{r \to 1} \sup_{a \in \mathbb{D}} \Phi_\phi(\alpha, p, s; a) = 0.$$  \hspace{1cm} (5.2)

Proof. Let $C_\phi : B^*_\alpha, \log \rightarrow F^*_p(\log, p, \alpha p - 2, s)$ be compact. This means that $\phi \in F^*_p(\log, p, \alpha p - 2, s)$.

Let

$$U^1_r = \{z : |\phi(z)| > r, r \in (0, 1)\},$$
and

\[ U_r^2 = \{ z : |\phi(z)| \leq r, r \in (0, 1) \}. \]

Let \( f_n(z) = \frac{z^n}{n} \) if \( \alpha \in [0, \infty) \) or \( f_n(z) = \frac{n}{z^n} \) if \( \alpha \in (0, 1) \). Without loss of generality, we only consider \( \alpha \in (0, 1) \). Since \( \| f_n \|_{B_n} \leq M \) and \( f_n(z) \to 0 \) as \( n \to \infty \), locally uniformly on the unit disk, then \( \| C_\phi(f_n) \|_{F_{log}^*(p, \alpha p - 2, s)} \to 0 \). This means that for each \( r \in (0, 1) \) and for all \( \varepsilon > 0 \), there exist \( N \in \mathbb{N} \) so that if \( n \geq N \), then

\[
\frac{N^{\alpha p}}{r^{p(1-N)}} \sup_{a \in \mathbb{D}} \int_{U_r^2} |\phi'(z)|^p(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon.
\]

If we choose \( r \) so that \( \frac{N^{\alpha p}}{r^{p(1-N)}} = 1 \), then

\[
\sup_{a \in \mathbb{D}} \int_{U_r^2} |\phi'(z)|^p(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon. \tag{5.3}
\]

Let now \( f \) be with \( \| f \|_{B_n} \leq 1 \). We consider the functions \( f_t(z) = f(tz), t \in (0, 1), f_t \to f \) uniformly on compact subset of the unit disk as \( t \to 1 \) and the family \( (f_t) \) is bounded on \( B_n \), thus

\[
\|(f_t \circ \phi) - (f \circ \phi)\| \to 0.
\]

Due to compactness of \( C_\phi \), we get that for \( \varepsilon > 0 \) there is \( t \in (0, 1) \) so that

\[
\sup_{a \in \mathbb{D}} \int_{U_r^2} |F_t(\phi(z))|^p(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon,
\]

where

\[
F_t(\phi(z)) = [(f \circ \phi)^* - (f_t \circ \phi)^*].
\]

Thus, if we fix \( t \), then

\[
\sup_{a \in \mathbb{D}} \int_{U_r^2} ((f \circ \phi)^*(z))^p(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s dA(z)
\leq 2^p \sup_{a \in \mathbb{D}} \int_{U_r^2} |F_t(\phi(z))|^p(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s dA(z)
+ 2^p \sup_{a \in \mathbb{D}} \int_{U_r^2} ((f_t \circ \phi)^*(z))^p(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s dA(z)
\leq 2^p \varepsilon + ||f_t^*||_{H^\infty} \sup_{a \in \mathbb{D}} \int_{U_r^2} |\phi'(z)|^p(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s dA(z)
\leq 2^p \varepsilon + 2^p ||f_t^*||_{H^\infty}^p.
i.e.,
\[
\sup_{a \in \mathbb{D}} \ell^p(a) \int_{U_1^a} ((f \circ \phi)^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z) \\
\leq 2^p \varepsilon (1 + ||f^*||^p_{L^p(\alpha, p - 2, s)}),
\]  
(5.4)
where we have used (4). On the other hand, for each \( ||f||_{g_{\alpha, \log}^*} \leq 1 \) and \( \varepsilon > 0 \), there exists a \( \delta \) depending on \( f \) and \( \varepsilon \), so that for \( r \in [\delta, 1) \),
\[
\sup_{a \in \mathbb{D}} \ell^p(a) \int_{U_1^a} ((f \circ \phi)^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon. 
\]  
(5.5)
Since \( C_{\phi} \) is compact, then it maps the unit ball of \( B_{\alpha, \log}^* \) to a relatively compact subset of \( F_{\log}^*(p, q, s) \). Thus, for each \( \varepsilon > 0 \), there exists a finite collection of functions \( f_1, f_2, ..., f_n \) in the unit ball of \( B_{\alpha, \log}^* \) so that for each \( ||f||_{g_{\alpha, \log}^*} \), there is \( k \in \{1, 2, 3, ..., n\} \) so that
\[
\sup_{a \in \mathbb{D}} \ell^p(a) \int_{U_1^a} |F_k(\phi(z))|^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon,
\]
where
\[ F_k(\phi(z)) = [(f \circ \phi)^* - (f_k \circ \phi)^*]. \]
Also, using (5), we get for \( \delta = max_{1 \leq k \leq n} \delta(f_k, \varepsilon) \) and \( r \in [\delta, 1) \), that
\[
\sup_{a \in \mathbb{D}} \ell^p(a) \int_{U_1^a} ((f_k \circ \phi)^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon.
\]
Hence, for any \( f, ||f||_{g_{\alpha, \log}^*} \leq 1 \), combining the two relations as above, we get the following
\[
\sup_{a \in \mathbb{D}} \ell^p(a) \int_{U_1^a} ((f \circ \phi)^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z) \leq 2^p \varepsilon.
\]
Therefore, we get that (2) holds. For the sufficiency, we use that \( \phi \in F_{\log}^*(p, \alpha p - 2, s) \) and (2) holds.

Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of functions in the unit ball of \( B_{\alpha, \log}^* \) so that \( f_n \rightarrow 0 \) as \( n \rightarrow \infty \), uniformly on the compact subsets of the unit disk. Let also \( r \in (0, 1) \). Then,
\[
||f_n \circ \phi||_{F_{\log}^*(p, \alpha p - 2, s)}^p \leq 2^p |f_n(\phi(0))| \\
+ 2^p \sup_{a \in \mathbb{D}} \ell^p(a) \int_{U_1^a} ((f_n \circ \phi)^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z) \\
+ 2^p \sup_{a \in \mathbb{D}} \ell^p(a) \int_{U_1^a} ((f_n \circ \phi)^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z) \\
= 2^p (I_1 + I_2 + I_3).
\]
Since \( f_n \to 0 \) as \( n \to \infty \), locally uniformly on the unit disk, then \( I_1 = |f_n(\phi(0))| \) goes to zero as \( n \to \infty \) and for each \( \varepsilon > 0 \), there is \( N \in \mathbb{N} \) so that for each \( n > N \),

\[
I_1 = |f_n(\phi(0))| \to 0 \quad \text{as} \quad n \to \infty \quad \text{and for each} \quad \varepsilon > 0,
\]

there is \( N \in \mathbb{N} \) so that for each \( n > N \),

\[
I_2 = \sup_{a \in D} \ell^p(a) \int_{U_2^1} ((f_n \circ \phi)^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z)
\]

\[\leq \varepsilon \|\phi\|_{F^*_{\log}(p, \alpha p - 2, s)}^p.\]

We also observe that

\[
I_3 = \sup_{a \in D} \ell^p(a) \int_{U_1^1} ((f_n \circ \phi)^*(z))^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z)
\]

\[\leq \|f\|_{B_{n, \log}}^p \times \sup_{a \in D} \ell^p(a) \int_{U_1^1} |\phi'(z)|^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s \left( (1 - |\phi(z)|^2)^p \left( \frac{2}{\log \left( \frac{2}{1 - |\phi(z)|^2} \right)} \right) \right)^s dA(z).\]

Under the assumption that (2) holds, then for every \( n > N \) and for every \( \varepsilon > 0 \), there exists \( r_1 \) so that for every \( r > r_1 \), \( I_3 < \varepsilon \).

Thus, if \( \phi(z) \in F^*_{\log}(p, \alpha p - 2, s) \), we get

\[
\|f_n \circ \phi\|_{F^*_{\log}(p, \alpha p - 2, s)}^p \leq 2^p \left( 0 + \varepsilon \|\phi\|_{F^*_{\log}(p, \alpha p - 2, s)}^p + \varepsilon \right) \leq C\varepsilon.
\]

Combining the above, we get \( \|C_{\phi}(f_n)\|_{F^*_{\log}(p, \alpha p - 2, s)}^p \to 0 \) as \( n \to \infty \) which proves compactness. Thus, the theorem we presented is proved.

6 Conclusions

We have obtained some essential and important \( D \)-metric spaces. Moreover, the important properties for \( D \)-metric on \( B^*_{n, \log} \) and \( F^*_{\log}(p, q, s) \) are investigated in Section 4. Finally, we introduced composition operators in hyperbolic weighted family of function spaces.

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