A linear time algorithm for minimum equitable dominating set in trees

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Abstract:

Let $G = (V, E)$ be a graph. A subset D^e of V is said to be an equitable dominating set if for every $v \in V \setminus D^e$ there exists $u \in D^e$ such that $uv \in E$ and $|\deg(u) - \deg(v)| \leq 1$, where, $\deg(u)$ and $\deg(v)$ denote the degree of the vertices u and v respectively. An equitable dominating set with minimum cardinality is called the minimum equitable dominating set and its cardinality is called the equitable domination number and it is denoted by γ^e . The problem of finding minimum equitable dominating set in general graphs is NP-complete. In this paper, we give a linear time algorithm to determine minimum equitable dominating set of a tree.

Keywords: Equitable domination; Linear time algorithm; Trees.

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1. Introduction

Domination problems are one of the extensively studied problems in graph theory as those can be applied to solve numerous problems in other fields of knowledge [1]. Let $G = (V, E)$ be a graph. A subset D of V is said to be a dominating set of G if every vertex is in D or is adjacent to some vertex in D , i.e., $v \in V$ implies that $v \in D$ or there exists $u \in D$ such that $uv \in E$. A dominating set of G with minimum cardinality is called the domination number of G and is denoted by $\gamma(G)$, or simply by γ .

A lots of variations of the domination problems are available in the literature. The concept of degree equitable domination was introduced by Prof. E. Sampathkumar. A subset D^e of V is said to be an equitable dominating set if for every $v \in V \setminus D^e$ there exists $u \in D^e$ such that $uv \in E$ and $|\deg(u) - \deg(v)| \leq 1$, where, $\deg(u)$ and $\deg(v)$ denote the degree of the vertices u and v respectively. We call a vertex $u \in V$ to be equitably dominated by another vertex $v \in D^e$ if $uv \in E$ and $|\deg(u) - \deg(v)| \leq 1$. It is obvious that every vertex equitably dominates itself. An equitable dominating set with minimum cardinality is called the minimum equitable dominating set and its cardinality is called the equitable domination number. Equitable domination number of the graph G is denoted by $\gamma^e(G)$. If there is no ambiguity, simply γ^e is also used to denote it.

Swaminathan and Dharmalingam [2] determined equitable domination number of some basic graphs like complete graph, path, cycle, wheel graph etc. Later, Sugumaran et al. [3] determined equitable domination numbers of some other graphs such as lollipop $L_{m,n}$, butterfly $BF(m, n)$, jellyfish $J(m, n)$ and subdivision of jellyfish $S(J(m, n))$.

It is well known that domination problem is an NP-complete problem in general graphs, but efficient algorithms are available for special classes of graphs [4]. In this paper, we give a linear time algorithm to obtain a minimum equitable dominating set and hence to find the equitable domination number of a tree.

2. Preliminaries

Since every equitable dominating set is also a dominating set but the converse is not true in general, it is easy to follow that for any graph G of order n , $1 \leq \gamma(G) \leq \gamma^e(G) \leq n$.

In this section, we present some basic results involving γ^e .

Proposition 1. [2]

- (i) For the complete graph K_n on n vertices, $\gamma^e(K_n) = 1$.
- (ii) For the paths P_n and the cycles C_n on n vertices, $\gamma^e(P_n) = \gamma^e(C_n) = \lceil \frac{n}{3} \rceil$.
- (iii) If W_n denotes the wheel on n vertices, then

$$\gamma^e(W_n) = \begin{cases} 1, & \text{if } n = 4, 5 \\ \lceil \frac{n-1}{3} \rceil + 1, & \text{otherwise.} \end{cases}$$
- (iv) For the complete bipartite graph $K_{m,n}$,

$$\gamma^e(K_{m,n}) = \begin{cases} 2, & \text{if } |m - n| \leq 1 \\ m + n, & \text{if } |m - n| \geq 2. \end{cases}$$

In the following, we give some results on equitable domination numbers of some fundamental tree structures.

Proposition 2. (i) For the star graph $K_{1,n-1}$ on n vertices,

$$\gamma^e(K_{1,n-1}) = \begin{cases} 1, & \text{if } n = 2, 3 \\ n, & \text{if } n > 3. \end{cases}$$

- (ii) A double star $DS_{m,n}$ on $m + n$ vertices is the graph obtained by joining the centers of two stars $K_{1,n-1}$ and $K_{1,m-1}$. If $m, n \geq 3$, then

$$\gamma^e(DS_{m,n}) = \begin{cases} m + n - 1, & \text{if } |m - n| \leq 1 \\ m + n, & \text{if } |m - n| \geq 2. \end{cases}$$

- (iii) The broom graph $B_{n,d}$ consists of a path P_d with d vertices, together with $n - d$ pendant vertices all adjacent to the same end vertex of P_d . If $n - d \geq 3$, then $\gamma^e(B_{n,d}) = (n - d + 1) + \lceil \frac{d-1}{3} \rceil$.

Proof.

- (i) Follows from Proposition 1 [(ii) and (iv)].
- (ii) The degree of two center vertices in $DS_{m,n}$ are m and n respectively. If the degree difference of the center vertices is greater than 1, i.e., $|m - n| \geq 2$ then, none of the vertices of $K_{1,m-1}$ can equitably dominate any vertex of $K_{1,n-1}$ in $DS_{m,n}$ and vice versa. Hence, the equitable domination number of $DS_{m,n}$ is the sum of the equitable domination number of each of the stars $K_{1,n-1}$ and $K_{1,m-1}$ and hence $\gamma^e(DS_{m,n}) = \gamma^e(K_{1,m-1}) + \gamma^e(K_{1,n-1}) = m + n$, since $m, n \geq 3$.

If the degree difference of the center vertices is less than or equal to 1, i.e., $|m-n| \leq 1$ then, one of the centers equitably dominates the other and hence $\gamma^e(DS_{m,n}) = \gamma^e(K_{1,m-1}) + \gamma^e(K_{1,n-1}) - 1 = m + n - 1$, since $m, n \geq 3$.

- (iii) Let $B_{n,d} = \{u_1, u_2, \dots, u_d, u_{d+1}, \dots, u_n\}$ be a broom graph on n vertices consisting of a path $P_d = \{u_1, u_2, \dots, u_d\}$ with d vertices, together with $n-d$ pendant vertices $\{u_{d+1}, u_{d+2}, \dots, u_n\}$. Without loss of generality, let us assume that all the pendant vertices are adjacent to the same end vertex u_d of P_d .

Now $\deg(u_d) = n - d + 1$. Since $n - d \geq 3$, u_d can neither equitably dominate any of its neighbours, nor it is equitably dominated by any other vertex in the graph. Hence all of the vertices $\{u_d, u_{d+1}, u_{d+2}, \dots, u_n\}$ will be present in the minimum equitable dominating set of $B_{n,d}$. Again, the vertices $\{u_1, u_2, \dots, u_{d-1}\}$ forms a path whose equitable domination number is $\lceil \frac{d-1}{3} \rceil$. Hence the equitable domination number of broom graph $B_{n,d}$ is given by $\gamma^e(B_{n,d}) = (n - d + 1) + \lceil \frac{d-1}{3} \rceil$.

□

Remark 1. It may be noted that –

- (i) for every $n \in N$, $\gamma(K_{1,n-1}) = 1$ and hence $\gamma^e(K_{1,n-1}) \neq \gamma(K_{1,n-1})$ for $n \geq 3$;
- (ii) for all $m, n \in N$, $\gamma(DS_{m,n}) = 2$ as the center vertices can dominate all other vertices and hence $\gamma^e(DS_{m,n}) \neq \gamma(DS_{m,n})$ for $m, n \geq 3$;
- (iii) for all $n, d \in N$, $\gamma(B_{n,d}) = 1 + \lceil \frac{d-2}{3} \rceil$ as u_d can dominate all its neighbours and hence $\gamma^e(B_{n,d}) \neq \gamma(B_{n,d})$ for $n - d \geq 3$.

3. The Algorithm

In this section, we describe an algorithm to find a minimum equitable dominating set and hence computing the equitable domination number of a tree T .

3.1. Outline of the algorithm

Let T be a tree on n vertices. We choose an arbitrary vertex as the root of T and label it by 1. Following a breadth first search (BFS), the vertices of

T are labelled as $1, 2, \dots, n$. Thus the tree T is uniquely represented by an array $P(i), i = 1, 2, \dots, n$ where $P(i)$ denotes the parent of the vertex i for $i = 2, 3, \dots, n$ and we let $P(1) = 0$.

As in the minimum dominating set algorithm of trees, given by Cockayne et al. [5], we also classify the vertices into three categories: required vertex (which must be in D^e — given label R), bound vertex (which are dominated by some vertex in D^e — given label B), and free vertex (which are not required vertices, but may be present in D^e — given label F). Initially, we label each vertex as a bound vertex. Then, starting from the pendent vertices, we go on changing possibly the labels of the parent vertices.

It is clear that for every i , either i , or $P(i)$, or both will be required vertices. If $|deg(i) - deg(P(i))| \leq 1$, then $P(i)$ will be a required vertex while i will remain a bound vertex. But if $|deg(i) - deg(P(i))| > 1$, the i will be a required vertex while $P(i)$ will be a required vertex, bound vertex or free vertex according to the degree of $P(P(i))$.

If i be a required vertex and $|deg(i) - deg(P(i))| \leq 1$, then $P(i)$ will be a free vertex. But if i be a required vertex and $|deg(i) - deg(P(i))| > 1$ then $P(i)$ remains bound vertex. But it may be changed later according to the value of $|deg(P(i)) - deg(P(P(i)))|$.

We give the pseudocode of the algorithm in the following subsection.

3.2. Pseudocode

Algorithm 1: Algorithm EQDOM

Input : A tree represented by an array $P(i)$ for each of the vertices $i = 1, 2, \dots, n$.
Output: A minimum equitable dominating set D^e .
Initialize: $D^e = \emptyset$;
for $i = 1 : n$ **do**
 $label(i) = B$;
 if $i = 1$ **then**
 $deg(i) = 0$;
 else
 $deg(i) = 1$;
 end
end
for $i = n : 2 : -1$ **do**
 $deg(P(i)) = deg(P(i)) + 1$;
end
for $i = n : 2 : -1$ **do**
 if $label(i) = B$ and $|deg(i) - deg(P(i))| \leq 1$ **then**
 $label(P(i)) = R$;
 $D^e = D^e \cup \{P(i)\}$;
 else if $label(i) = B$ and $|deg(i) - deg(P(i))| > 1$ **then**
 $label(i) = R$;
 $D^e = D^e \cup \{i\}$;
 if $label(i) = R$ and $|deg(i) - deg(P(i))| \leq 1$ **then**
 $label(P(i)) = F$;
 end
 end
end
if $label(1) = B$ or $label(1) = R$ **then**
 $D^e = D^e \cup \{1\}$;
end

3.3. Illustrative example

Let T be a BFS tree (shown in Figure 1) with vertices $V(T) = \{1, 2, 3, \dots, 18\}$ and parent of each vertex is given in second row of Table 3.1. Initially, all vertices are labeled as bound (B) vertex. Now, we traverse the vertices of T in descending order from 18 to 1 to update the vertex labels (whether it is bound, required or free). Gradually, the minimum cardinality equitable dominating set will also be updated in each step as elaborated below:

Figure 1: T be a tree of order 18

$i = 18$; $P(i) = 10$; $|deg(i) - deg(P(i))| \leq 1$ hence $label(P(i)) = R$ and $D^e = \{10\}$

$i = 17$; $P(i) = 9$; $|deg(i) - deg(P(i))| > 1$ hence $label(i) = R$ and $D^e = \{10, 17\}$

$i = 16$; $P(i) = 9$; $|deg(i) - deg(P(i))| > 1$ hence $label(i) = R$ and $D^e = \{10, 17, 16\}$

$i = 15$; $P(i) = 8$; $|deg(i) - deg(P(i))| \leq 1$ hence $label(P(i)) = R$ and $D^e = \{10, 17, 16, 8\}$

$i = 14$; $P(i) = 7$; $|deg(i) - deg(P(i))| \leq 1$ hence $label(P(i)) = R$ and $D^e = \{10, 17, 16, 8, 7\}$

$i = 13$; $P(i) = 6$; $|deg(i) - deg(P(i))| \leq 1$ hence $label(P(i)) = R$ and $D^e = \{10, 17, 16, 8, 7, 6\}$

$i = 12$; $P(i) = 5$; $|deg(i) - deg(P(i))| > 1$ hence $label(i) = R$ and $D^e = \{10, 17, 16, 8, 7, 6, 12\}$

$i = 11$; $P(i) = 5$; $|deg(i) - deg(P(i))| > 1$ hence $label(i) = R$ and $D^e = \{10, 17, 16, 8, 7, 6, 12, 11\}$

$i = 10$; $P(i) = 4$; $label(i) = R$; $|deg(i) - deg(P(i))| \leq 1$ hence $label(P(i)) = F$
 and $D^e = \{10, 17, 16, 8, 7, 6, 12, 11\}$
 $i = 9$; $P(i) = 3$; $|deg(i) - deg(P(i))| \leq 1$ hence $label(P(i)) = R$ and
 $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3\}$
 $i = 8$; $P(i) = 3$; $label(i) = R$; $label(P(i)) = R$ hence
 $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3\}$
 $i = 7$; $P(i) = 2$; $label(i) = R$; hence $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3\}$
 $i = 6$; $P(i) = 2$; $label(i) = R$ hence $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3\}$
 $i = 5$; $P(i) = 2$; $label(i) = B$; $|deg(i) - deg(P(i))| \leq 1$ hence $label(P(i)) = R$
 and $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3, 2\}$
 $i = 4$; $P(i) = 1$; $label(i) = F$ hence $label(i) = B$ and
 $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3, 2\}$
 $i = 3$; $P(i) = 1$; $label(i) = R$ hence $label(P(i)) = F$ and
 $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3, 2\}$
 $i = 2$; $P(i) = 1$; $label(i) = R$ hence $label(P(i)) = F$ and
 $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3, 2\}$
 $i = 1$; $P(i) = 0$; $label(i) = F$ hence $D^e = \{10, 17, 16, 8, 7, 6, 12, 11, 3, 2\}$.

i	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$P(i)$	10	9	9	8	7	6	5	5	4	3	3	2	2	2	1	1	1	0
$deg(i)$	1	1	1	1	1	1	1	1	2	3	2	2	2	3	2	3	4	3
$label(i)$	B	R	R	B	B	B	R	R	R	B	R	R	R	B	B	R	R	F

Table 3.1: Label of different vertices of T .

Therefore, $D^e(T) = \{17, 16, 12, 11, 10, 8, 7, 6, 3, 2\}$ and hence $\gamma^e(T) = 10$.

3.4. Correctness and complexity of the algorithm

Lemma 1. Let T be a BFS tree in which p be a pendent vertex and $q = P(p)$ and $D^e(T)$ be the minimum equitable dominating set. Then p must be in $D^e(T)$ if $deg(q) \geq 3$.

Proof. Since p can be dominated by no other vertices except q and p itself, p must be in $D^e(T)$ as $|deg(q) - deg(p)| \geq 1$ and so q can not equitably dominate p . $\square$

Lemma 2. *Let T be a BFS tree in which p be any vertex other than the root and $q = P(p)$. Then an equitable dominating set $D_p^e(T)$ containing p can not have the cardinality less than another equitable dominating set $D_q^e(T)$ containing q .*

Proof. If p is a pendent vertex, then the result is obvious, since q can dominate possibly p and $P(q)$ both, whereas p can never dominate $P(q)$.

Suppose, children of p are all pendent vertices. Then p can dominate equitably all its children only when it has only one child, because otherwise $\deg(p) \geq 3$ and it can not equitably dominate any pendent vertex.

If neither p is a pendent vertex, nor all its children are pendent vertices, the problem can be reduced to subproblems of either of the above two types. Combining them, the lemma follows. $\square$

Theorem 1. *The algorithm EqDom correctly yields a minimum equitable dominating set of a BFS tree T in linear time.*

Proof. Correctness of the algorithm follows from Lemma 1 and 2.

Since the algorithm requires an initial assignment of $|V(T)|$ values and scanning of the linear array $P(\cdot)$ twice, complexity of the algorithm is linear. $\square$

4. Concluding Remarks

In this paper, we have obtained a linear time algorithm to find a minimum equitable dominating set of a tree. We have also illustrated the algorithm through an example. Correctness and complexity of the algorithm are also established. As future scope of works, developing linear or efficient polynomial time algorithm for other classes of graphs, such as interval graph, permutation graph, chordal graph, etc. may be considered.

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