Unicyclic graphs with equal domination and complementary tree domination numbers

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Abstract

Let $G = (V, E)$ be a simple graph. A set $D \subseteq V(G)$ is a dominating set if every vertex in $V(G) \setminus D$ is adjacent to a vertex of $D$. A dominating set $D$ of a graph $G$ is a complementary tree dominating set if induced sub graph $(V \setminus D)$ is a tree. The domination (complementary tree domination, respectively) number of $G$ is the minimum cardinality of a dominating (complementary tree dominating, respectively) set of $G$. We characterize all unicyclic graphs with equal domination and complementary tree domination numbers.

Keywords : Domination, complementary tree domination, unicyclic graphs.

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1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex $v$ of $G$ we mean the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The degree of a vertex $v$, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We denote $L(G)$ to be the set of leaves of the graph $G$ and $S(G)$ is the set of all support vertices of $G$. The path on $n$ vertices we denote by $P_n$. Let $T$ be a tree, and let $v$ be a vertex of $T$. We say that $v$ is adjacent to a path $P_n$ if there is a neighbor of $v$, say $x$, such that the subtree resulting from $T$ by removing the edge $vx$ and which contains the vertex $x$ as a leaf, is a path $P_n$.

A subset $D \subseteq V(G)$ is a dominating set of $G$ if every vertex of $V(G) \setminus D$ has a neighbor in $D$, while it is a complementary tree dominating set, abbreviated CTDS, of $G$ if the induced sub graph $(V \setminus D)$ is a tree. The domination (complementary tree domination, respectively) number of a graph $G$, denoted by $\gamma(G)$ ($\gamma_{ctd}(G)$, respectively), is the minimum cardinality of a dominating (complementary tree dominating, respectively) set of $G$. A complementary tree dominating set of $G$ of minimum cardinality is called a $\gamma_{ctd}(G)$-set. The complementary tree domination in graphs was studied in [5]. For a comprehensive survey of domination in graphs, see [1, 2].

A unicyclic graph is a graph that contains exactly one cycle. In this paper we provide a constructive characterization of all unicyclic graphs with equal domination number and complementary tree domination number. In [3], unicyclic graphs with equal total and total outer-connected domination numbers are characterized.

2. Preliminary results

We begin with the following straightforward observation.

Observation 1. [5] Every leaf of a graph $G$ is in every $\gamma_{ctd}(G)$-set.

In [4] trees with equal domination number and complementary tree domination numbers are characterized. For this purpose the family $T$ of trees $T = T_k$ is defined. Let $T_1$ be a path $P_4$. If $k$ is a positive integer, then $T_{k+1}$ can be obtained recursively from $T_k$ by one of the following operations.
• Operation $O_1$: Attach a path $P_2$ by joining its any vertex to a vertex of $T_k$, which is not a leaf and is adjacent to a support vertex of degree two.

• Operation $O_2$: Attach a path $P_2$ by joining its any vertex to a support vertex of $T_k$.

**Theorem 1.** [4] Let $T$ be a tree. Then $\gamma_{ctd}(T) = \gamma(T)$ if and only if $T \in \mathcal{T}$.

3. Unicyclic graphs

We characterize all connected unicyclic graphs for which $\gamma(G) = \gamma_{ctd}(G)$. To this, we define $\mathcal{C}$ to be the family of all graphs $G$ for which exists a tree $T$ belonging to the family $\mathcal{T}$, such that $G$ is obtained from $T$ by the operation:

**Operation $B$:** Let $u, v$ be any two support vertices of $T$. Let $x$ and $y$ be the leaves adjacent to $u$ and $v$, respectively. Identify $x$ with $y$.

Let us also assume that $C_3$ and $C_4$ belong to $\mathcal{C}$ and observe that $C_3$ is obtained from $P_4 \in \mathcal{T}$ by the above operation.

**Lemma 2.** If $G$ belong to the family $\mathcal{C}$, then $\gamma(G) = \gamma_{ctd}(G)$.

**Proof.** If $G$ is a cycle belonging to $\mathcal{C}$, then the result is immediate. Let us now assume that $G$ is obtained from a tree $T \in \mathcal{T}$ by Operation $B$. Let $G$ be obtained from $T$ by identifying the leaves $x$ and $y$. Denote by $w$ the vertex obtained by identifying $x$ and $y$. It is easy to see that $L(G) \cup \{w\}$ is a minimum dominating set of $G$. Thus $\gamma(G) = |L(G)| + 1$. On the other hand, $L(G) \cup \{w\}$ is a complementary tree dominating set of $G$. Thus we have $|L(G)| + 1 = \gamma(G) \leq \gamma_{ctd}(G) \leq |L(G)| + 1$. Thus we have $\gamma(G) = \gamma_{ctd}(G)$. $\Box$

**Lemma 3.** If $G$ is a connected unicyclic graph with $\gamma(G) = \gamma_{ctd}(G)$, then $G$ belongs to family $\mathcal{C}$.

**Proof.** Let $G$ be a connected unicyclic graph, where $C_k = (v_1, v_2, v_3, \ldots, v_k)$ is the unique cycle of $G$. Assume first that each vertex of $C_k$ is of degree 2. Then $G$ is a cycle $C_k$ for some $k \geq 3$. It is clear that $\gamma_{ctd}(C_k) = k - 2$ for $k \geq 3$. On the other hand, $\gamma(C_k) < k - 2$ for $k \geq 5$. Thus $\gamma(C_k) = \gamma_{ctd}(C_k)$ if $k \in \{3, 4\}$. 
Assume that $G$ is not a cycle. If $v_i \in V(C_k)$, then let $T(v_i)$ be the tree obtained from $G$ by removing edges $v_i v_{i+1}$ and $v_{i-1} v_i$ (where the indices are taken modulo $k$ added 1) and containing $v_i$. Let $v_i$ be the root of $T(v_i)$. Let $D_{ctd}$ be a minimum complementary tree dominating set of $G$.

Assume without loss of generality, that $d_G(v_1) \geq 3$, and denote by $x$ any element of $V(T(v_1))$ which is neither a leaf nor a support vertex. Let $x \in D_{ctd}$. Then either $V(G) \setminus D_{ctd} \subseteq V(T(x))$ or $V(G) \setminus D_{ctd} \subseteq V(G) \setminus V(T(x))$. Let $V(G) \setminus D_{ctd} \subseteq V(G) \setminus V(T(x))$. It is clear that $V(G) \setminus D_{ctd}$ contains a cycle, contradiction to the definition of $D_{ctd}$. Now assume $V(G) \setminus D_{ctd} \subseteq V(T(x))$. Then $D_{ctd} \setminus \{u\}$ where $u$ is a leaf in $T(x)$ is a dominating set of $G$ of smaller cardinality than $\gamma(G)$, a contradiction. Hence, we conclude that every vertex in $T(v_1)$ is either a support vertex or a leaf.

Assume $V(C) \cap D_{ctd} = \emptyset$. The complement of $D_{ctd}$ contains a cycle, a contradiction. Now assume, without loss of generality, that $v_1 \in V(C) \cap D_{ctd}$ and $d_G(v_1) \geq 3$. Then obviously $V(T(v_1)) \subseteq D_{ctd}$. It is easy to see that $D_{ctd} \setminus \{u\}$ where $u$ is a leaf in $T(v_1)$ is a dominating set of $G$ of smaller cardinality than $\gamma(G)$, a contradiction. Hence, $d_G(v_1) = 2$.

Now assume $d_G(v_2) \geq 3$ and $d_G(v_k) \geq 3$. Suppose $v_2$ and $v_k$ is in $D_{ctd}$, then $D_{ctd} \setminus \{v_1\}$ is a dominating set of cardinality smaller than $\gamma(G)$, a contradiction. Without loss of generality, assume $v_2 \in D_{ctd}$. Arguing as in the previous case, we get $d_G(v_2) = 2$. Assume that $v_3$ and $v_k$ not in $D_{ctd}$. Then since $D_{ctd}$ is a complementary tree dominating set, exactly two vertices of $V(C_k)$ belong to $D_{ctd}$, namely $v_1$ and $v_2$. It is easy to see that $T(v_1) \setminus \{v_i\} \in D_{ctd}, 3 \leq i \leq k$. It is obvious that $D_{ctd} \setminus \{v_2\}$ is a dominating set of cardinality smaller than $\gamma(G)$, a contradiction. Thus $v_1$ is the only vertex in $D_{ctd}$ set of $G$.

Denote by $G_1$ the graph obtained from $G$ by removing the edge $v_1 v_2$ and attaching the vertex $x$ to the vertex $v_2$. It is obvious that $\gamma(G) \leq \gamma(G_1)$. Suppose $D_{ctd}$ is a $\gamma_{ctd}(G_1)$-set of cardinality smaller than $\gamma_{ctd}(G_1)+1$. The vertex $x$ is a leaf in $G_1$. By observation 1, the leaf $x \in D_{ctd}$. Then $D'_{ctd} = D_{ctd} \setminus \{x\}$ is a complementary tree dominating set of $G$. Thus $\gamma(G) = \gamma_{ctd}(G) \leq |D'_{ctd}| = \gamma_{ctd}(G_1) + 1$. It is easy to observe that $D'_{ctd} \cup \{x\}$ is a complementary tree dominating set of $G_1$, so $\gamma_{ctd}(G_1) \leq \gamma_{ctd}(G) + 1$. Equivalently, $\gamma_{ctd}(G) \geq \gamma_{ctd}(G_1) - 1$. This implies that $\gamma_{ctd}(G) = \gamma_{ctd}(G_1) - 1$. Since $G_1$ is a tree, theorem 1 implies that $G_1$ belongs to the family $T$. We conclude that $G$ is obtained from a tree belonging to the family $T$ by operation $B$. Therefore, $G$ belongs to the family $C$. □
References


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