An alternative proof of a Tauberian theorem for Abel summability method

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Abstract

Using a corollary to Karamata’s main theorem [Math. Z. 32 (1930), 319–320], we prove that if a slowly decreasing sequence of real numbers is Abel summable, then it is convergent in the ordinary sense.


Keywords : Abel summability, slowly decreasing sequences, Tauberian conditions and theorems.
1. Introduction

A number of authors such as Schmidt [9], Maddox [6], Móricz [8], and Talo and Başar [11] have proved several Tauberian theorems for some summability methods for which slowly decreasing condition for sequences is a Tauberian condition. Schmidt [9] obtained that the slowly decreasing condition for sequences of real numbers is a Tauberian condition for Abel summability. Maddox [6] introduced the slowly decreasing sequence in an ordered linear space and proved that a Cesàro summable sequence is convergent if it is slowly decreasing in an ordered linear space. Móricz [8] established a Tauberian theorem which states that ordinary convergence of a sequence follows from its statistical Cesàro summability if it is slowly decreasing. Talo and Başar [11] introduced the concept of slowly decreasing sequences for fuzzy numbers and they proved that the slowly decreasing condition for sequences is a Tauberian condition for the statistical convergence and Cesàro summability for sequences of fuzzy numbers.

Littlewood [5] proved that \( n(u_n - u_{n-1}) = O(1) \) is a Tauberian condition for Abel summability of \((u_n)\). But his proof was complicated and based on the repeated differentiation. A first clever and surprisingly simple proof based on Weierstrass approximation theorem of Littlewood’s theorem was given by Karamata [2].

The main purpose of this study is to give an alternative simpler proof of the following Tauberian theorem which is more general than Littlewood’s theorem [5] for Abel summability method.

**Theorem 1.1.** If \((u_n)\) is Abel summable to \(s\) and slowly decreasing, then \(\lim_n u_n = s\).

To prove Theorem 1.1, we first obtain Cesàro convergence of the generator sequence of a given sequence \((u_n)\) by means of a corollary to Karamata’s main Theorem, and then recover convergence of \((u_n)\) by Tauber’s second theorem [12].

Our proof is much easier than the existing one and uses the well known results in Tauberian theory. For a different proof of Theorem 1.1, see [1].
2. Preliminaries

For a sequence \( u = (u_n) \) of real numbers, we write \( (u_n) \) in terms of \( (v_n) \) as

\[
(2.1) \quad u_n = v_n + \sum_{k=1}^{n} \frac{v_k}{k} + u_0, \quad (n = 1, 2, \ldots)
\]

where \( v_n = \frac{1}{n+1} \sum_{k=1}^{n} k(u_k - u_{k-1}) \). The sequence \( (v_n) \) is called a generator sequence of \( (u_n) \). We note that \( \sigma_n^{(1)}(u) = \frac{1}{n+1} \sum_{k=0}^{n} u_k = u_0 + \sum_{k=1}^{n} \frac{v_k}{k} \).

Let \( u = (u_n) \) be a sequence of real numbers. For each nonnegative integer \( m \), we define \( \sigma_n^{(m)}(u) \) by

\[
\sigma_n^{(m)}(u) = \begin{cases} 
1 & \text{for } m = 0 \\
\frac{1}{n+1} \sum_{k=0}^{n} \sigma_k^{(m-1)}(u), & m \geq 1 \\
u_n, & m = 0
\end{cases}
\]

A sequence \( (u_n) \) is said to be Abel summable to \( s \) if \( u_0 + \sum_{n=1}^{\infty} (u_n - u_{n-1})x^n \) converges for \( 0 < x < 1 \), and tends to \( s \) as \( x \to 1^- \).

A sequence \( (u_n) \) is called \( (A,m) \) summable to \( s \) if \( (\sigma_n^{(m)}(u)) \) is Abel summable to \( s \). If \( m = 0 \), then \( (A,m) \) summability reduces to Abel summability. It is clear that Abel summability of \( (u_n) \) implies \( (A,m) \) summability of \( (u_n) \).

Throughout this work, the symbol \([\lambda n]\) denotes the integral part of the product \( \lambda n \).

A sequence \( (u_n) \) is said to be slowly decreasing \([9]\) if

\[
(2.2) \quad \lim_{\lambda \to 1^+} \liminf_{n \to \infty} \min_{n+1 \leq k \leq [\lambda n]} (u_k - u_n) \geq 0
\]

or equivalently \([8]\),

\[
(2.3) \quad \lim_{\lambda \to 1^-} \liminf_{n \to \infty} \min_{[\lambda n]+1 \leq k \leq n} (u_n - u_k) \geq 0.
\]

Notice that \( (u_n) \) is slowly decreasing if the classical one-sided Tauberian condition of Landau \([1]\) is satisfied, that is, there exists a positive constant \( C > 0 \) such that

\[
(2.4) \quad n(u_n - u_{n-1}) \geq -C
\]

for all nonnegative \( n \). Indeed, for any \( k > n \), we have

\[
u_k - u_n = \sum_{j=n+1}^{k} (u_j - u_{j-1}) \geq -C \sum_{j=n+1}^{k} \frac{1}{j} \geq -C \log \left( \frac{k}{n} \right)
\]
whence we conclude that
\[ \liminf_{n \to \infty} \min_{n+1 \leq k \leq \lfloor \lambda n \rfloor} (u_k - u_n) \geq -C \log \lambda, \quad \lambda > 1. \]

Taking \( \lambda \to 1^+ \), we have the inequality (2.2).

Note that we used \( C \) to denote a constant, possibly different at each occurrence.

A sequence \((u_n)\) is slowly increasing if and only if \((-u_n)\) is slowly decreasing, and an equivalent definition of a slowly increasing sequence as follows:

A sequence \((u_n)\) is said to be slowly increasing if
\[ \lim_{\lambda \to 1^+} \limsup_{n \to \infty} \max_{n+1 \leq k \leq \lfloor \lambda n \rfloor} (u_k - u_n) \leq 0. \tag{2.5} \]

The condition (2.5) is reformulated as follows (see [8]):
\[ \lim_{\lambda \to 1^-} \limsup_{n \to \infty} \max_{\lfloor \lambda n \rfloor + 1 \leq k \leq n} (u_n - u_k) \leq 0. \tag{2.6} \]

It is plain that a sequence \((u_n)\) is said to be slowly oscillating if and only if \((u_n)\) is both slowly increasing and slowly decreasing. Notice that each of the conditions (2.2) and (2.5) is necessary for convergence (see [4]).

If \((u_n)\) converges to \( s \), then \((u_n)\) is Abel summable to \( s \). However, the converse of this statement is not always true. Note that Abel summability of \((u_n)\) implies convergence of \((u_n)\) under certain additional hypotheses called Tauberian conditions. Any theorem which states that convergence of sequence \((u_n)\) follows from Abel summability of \((u_n)\) and some Tauberian condition(s) is called a Tauberian theorem for Abel summability method.

3. Corollary to Karamata’s Main Theorem and Lemmas

Our proof is based on the following corollary to Karamata’s main theorem and three Lemmas.

**Corollary to Karamata’s Main Theorem.** ([2]) If \( u = (u_n) \) is Abel summable to \( s \) and \( u_n \geq -C \) for some nonnegative \( C \), then \( \lim_{n} \sigma_n^{(1)}(u) = s \).

**Lemma 3.1.** ([3]) If, for \( x \to 1^- \), a function \( f(x) \), which is integrable in \([0, 1]\), satisfies the limiting relation
\[ (1 - x)^2 f(x) \to s, \tag{3.1} \]
then, for \( x \to 1^- \), we also have

\[ (1 - x) \int_0^x f(t) \, dt \to s. \]  

(3.2)

The next lemma gives a necessary condition for a slowly decreasing sequence in terms of the generator sequence \((v_n)\).

**Lemma 3.2.** ([7]) If \((u_n)\) is slowly decreasing, then \( v_n \geq -C \) for some \( C \), where \( v_n = \frac{1}{n+1} \sum_{k=1}^n k(u_k - u_{k-1}) \).

Next, we represent the difference \( u_n - \sigma_n^{(1)}(u) \) in two different ways.

**Lemma 3.3.** ([10]) Let \( u = (u_n) \) be a sequence of real numbers.

(i) For \( \lambda > 1 \) and sufficiently large \( n \),

\[
 u_n - \sigma_n^{(1)}(u) = \frac{\lceil \lambda n \rceil + 1}{\lceil \lambda n \rceil - n} \left( \sigma_{\lceil \lambda n \rceil}^{(1)}(u) - \sigma_n^{(1)}(u) \right) - \frac{1}{\lceil \lambda n \rceil - n} \sum_{k=n+1}^{\lceil \lambda n \rceil} (u_k - u_n).
\]

(ii) For \( 0 < \lambda < 1 \) and sufficiently large \( n \),

\[
 u_n - \sigma_n^{(1)}(u) = \frac{\lceil \lambda n \rceil + 1}{n - \lceil \lambda n \rceil} \left( \sigma_n^{(1)}(u) - \sigma_{\lceil \lambda n \rceil}^{(1)}(u) \right) + \frac{1}{n - \lceil \lambda n \rceil} \sum_{k=\lceil \lambda n \rceil + 1}^n (u_n - u_k).
\]

(3.3)

(3.4)

### 4. Proof of Theorem 1.1

**Proof.** Since \((u_n)\) is Abel summable to \( s \), then \((\sigma_n^{(1)}(u))\) is also Abel summable to \( s \). Hence, we conclude by (2.1) that \((v_n) = \left( \frac{1}{n+1} \sum_{k=0}^n k(u_k - u_{k-1}) \right)\) is Abel summable to zero by Lemma 3.1. It follows by Lemma 3.2 that there exists a nonnegative \( C \) such that

\[ v_n \geq -C. \]  

(4.1)

Taking (4.1) and the fact that \((v_n)\) is Abel summable to zero into account, we obtain by Corollary to Karamata’s Main Theorem that \( \sigma_n^{(1)}(v) = \)
$o(1)$ as $n \to \infty$. Since $(\sigma_n^{(1)}(u))$ is Abel summable to $s$ and $\sigma_n^{(1)}(v) = o(1)$ as $n \to \infty$, we have that $(\sigma_n^{(1)}(u))$ converges to $s$ by Tauber’s second theorem [12].

By the fact that every convergent sequence is slowly increasing, we have

$(\sigma_n^{(1)}(u))$ is slowly increasing. Thus, $(-\sigma_n^{(1)}(u))$ is slowly decreasing. Since $(s_n)$ is slowly decreasing, $(v_n)$ is slowly decreasing.

By Lemma 3.3 (i), we have

$$v_n - \sigma_n^{(1)}(v) = \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor - n} \left( \sigma_{\lfloor \lambda n \rfloor}^{(1)}(v) - \sigma_n^{(1)}(v) \right) - \frac{1}{\lfloor \lambda n \rfloor - n} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} (v_k - v_n).$$

(4.2)

It is easy to verify that for $\lambda > 1$ and sufficiently large $n$,

$$\frac{\lambda}{2(\lambda - 1)} \leq \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor - n} \leq \frac{3\lambda}{2(\lambda - 1)}.$$  

(4.3)

By $\sigma_n^{(1)}(v) = o(1)$ as $n \to \infty$ and (4.10), for all $\lambda > 1$,

$$\lim_{n \to \infty} \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor - n} \left( \sigma_{\lfloor \lambda n \rfloor}^{(1)}(v) - \sigma_n^{(1)}(v) \right) = 0.$$  

(4.4)

By (4.2) and (4.3), we have

$$v_n - \sigma_n^{(1)}(v) \leq \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor - n} \left( \sigma_{\lfloor \lambda n \rfloor}^{(1)}(v) - \sigma_n^{(1)}(v) \right) - \min_{n+1 \leq k \leq \lfloor \lambda n \rfloor} (v_k - v_n).$$  

(4.5)

Taking $\limsup$ of both sides of (4.5), we have

$$\limsup_n (v_n - \sigma_n^{(1)}(v)) \leq \limsup_{n \to \infty} \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor - n} \left( \sigma_{\lfloor \lambda n \rfloor}^{(1)}(v) - \sigma_n^{(1)}(v) \right) - \liminf_n \min_{n+1 \leq k \leq \lfloor \lambda n \rfloor} (v_k - v_n).$$  

(4.6)
The inequality (4.6) becomes

\[ \limsup_n (v_n - \sigma_n^{(1)}(v)) \leq - \liminf_n \min_{n+1 \leq k \leq [\lambda n]} (v_k - v_n) \]

by (4.4). Taking \( \lambda \to 1^+ \) in (4.7), we have

\[ \limsup_n (v_n - \sigma_n^{(1)}(v)) \leq 0 \]

by (2.2).

By Lemma 3.3 (ii), we have

\[ v_n - \sigma_n^{(1)}(v) = \frac{[\lambda n] + 1}{n - [\lambda n]} \left( \sigma_n^{(1)}(v) - \sigma_{[\lambda n]}^{(1)}(v) \right) + \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (v_n - v_k). \]

(4.9)

It is easy to verify that for \( 0 < \lambda < 1 \) and sufficiently large \( n \),

\[ \frac{\lambda}{2(1-\lambda)} \leq \frac{[\lambda n] + 1}{n - [\lambda n]} \leq \frac{3\lambda}{2(1-\lambda)}. \]

(4.10)

By \( \sigma_n^{(1)}(v) = o(1) \) as \( n \to \infty \) and (4.10), for all \( 0 < \lambda < 1 \),

\[ \lim_{n \to \infty} \frac{[\lambda n] + 1}{n - [\lambda n]} \left( \sigma_n^{(1)}(v) - \sigma_{[\lambda n]}^{(1)}(v) \right) = 0. \]

(4.11)

By (4.9) and (4.10), we have

\[ v_n - \sigma_n^{(1)}(v) \geq \frac{[\lambda n] + 1}{n - [\lambda n]} \left( \sigma_n^{(1)}(v) - \sigma_{[\lambda n]}^{(1)}(v) \right) + \min_{[\lambda n]+1 \leq k \leq n} (v_n - v_k). \]

(4.12)

Taking \( \liminf \) of both sides of (4.12), we have

\[ \liminf_n (v_n - \sigma_n^{(1)}(v)) \geq \liminf_n \frac{[\lambda n] + 1}{n - [\lambda n]} \left( \sigma_n^{(1)}(v) - \sigma_{[\lambda n]}^{(1)}(v) \right) \]

\[ + \liminf_n \min_{[\lambda n]+1 \leq k \leq n} (v_n - v_k). \]

(4.13)
The inequality (4.13) becomes

\[
\lim \inf_n (v_n - \sigma_n^{(1)}(v)) \geq \lim \inf_n \min_{|\lambda n| + 1 \leq k \leq n} (v_n - v_k)
\]

by (4.11).

Taking \( \lambda \to 1^- \) in (4.14), we have

\[
\lim \inf_n (v_n - \sigma_n^{(1)}(v)) \geq 0
\]

by (2.3).

Combining (4.8) and (4.15) yields that \( v_n = o(1) \) as \( n \to \infty \). Since \((u_n)\) is Abel summable to \( s \) and \( v_n = o(1) \) as \( n \to \infty \), \( \lim_n u_n = s \) by Tauber’s second theorem [12]. This completes the proof. \( \Box \)

Using Theorem 1.1, we show that slow decrease of \((u_n)\) is also a Tauberian condition for \((A,m)\) summability method.

**Theorem 4.1.** If \((u_n)\) is \((A,m)\) summable to \( s \) and slowly decreasing, then \( \lim_n u_n = s \).

**Proof.** Let \((u_n)\) be slowly decreasing. Then, we have \( v_n \geq -C \) for some \( C \) by Lemma 3.2. Since \( n(\sigma_n^{(1)}(u) - \sigma_{n-1}^{(1)}) = v_n \) for all nonnegative \( n \), we conclude that \((\sigma_n^{(1)}(u))\) is slowly decreasing if we replace \( u_n \) in (2.4) by \( \sigma_n^{(1)}(u) \).

It easily follows that \((\sigma_n^{(m)}(u))\) is slowly decreasing for each nonnegative \( m \).

Since \((u_n)\) is \((A,m)\) summable to \( s \), we have

\[
\lim_n \sigma_n^{(m)}(u) = s
\]

by Theorem 1.1. By definition, we have

\[
\sigma_n^{(m)}(u) = \sigma_n^{(1)}(\sigma_n^{(m-1)}(u)).
\]

From (4.16) and (4.17) it follows that \((u_n)\) is \((A,m-1)\) summable to \( s \). Since \((\sigma_n^{(m-1)}(u))\) is slowly decreasing, we have \( \lim_n \sigma_n^{(m-1)}(u) = s \) by Theorem 1.1. Continuing in this way, we obtain that \( \lim_n u_n = s \). \( \Box \)
References


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