The $t$-pebbling number of Jahangir graph $J_{3,m}$

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Abstract

The $t$-pebbling number, $f_t(G)$, of a connected graph $G$, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, $t$ pebbles can be moved to any specified target vertex by a sequence of pebbling moves, each move removes two pebbles of a vertex and placing one on an adjacent vertex. In this paper, we determine the $t$-pebbling number for Jahangir graph $J_{3,m}$ and finally we give a conjecture for the $t$-pebbling number of the graph $J_{n,m}$.

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1. Introduction

We begin by introducing relevant terminology and background on the subject. Here, the term graph refers to a simple graph. A function $\phi : V(G) \rightarrow N \cup \{0\}$ is called a pebbling. Let $\phi(v)$ denote the number of pebbles on the vertex $v$ and $\phi(V(A))$ denote the number of pebbles on the vertices of the subgraph $A$ of $G$. The quantity $\sum_{x \in V(G)} \phi(x)$ is called the size of $\phi$; the size of $\phi$ is just the total number of pebbles assigned to vertices. A pebbling move (step) consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. For a graph $G$, if $\phi$ is a distribution or configuration of pebbles onto the vertices of $G$ and it is possible to move a pebble to the target vertex $v$, then we say that $\phi$ is $v$-solvable; otherwise, $\phi$ is $v$-unsolvable. Then $\phi$ is solvable if it is $v$-solvable for all $v \in V(G)$, and unsolvable otherwise. If $\phi(v) = 1$ or $\phi(u) \geq 2$ where $uv \in E(G)$, then we can easily move one pebble to $v$. So, we always assume that $\phi(v) = 0$ and $\phi(u) \leq 1$ for all $uv \in E(G)$ when $v$ is the target vertex.

The $t$-pebbling number of a vertex $v$ in a graph $G$, $f_t(v,G)$, is the smallest positive integer $m$ such that however $m$ pebbles are placed on the vertices of the graph, $t$ pebbles can be moved to $v$ in finite number of pebbling moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. The $t$-pebbling number of $G$, $f_t(G)$, is defined to be the maximum of the pebbling numbers of its vertices. Thus the $t$-pebbling number of a graph $G$, $f_t(G)$, is the least $m$ such that, for any configuration of $m$ pebbles to the vertices of $G$, we can move $t$ pebbles to any vertex by a sequence of moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the pebbling number of $G$.

Fact 1. ([12]) For any vertex $v$ of a graph $G$, $f(v,G) \geq n$ where $n = |V(G)|$.

Fact 2. ([12]) The pebbling number of a graph $G$ satisfies

$$f(G) \geq \max\{2^{\text{diam}(G)}, |V(G)|\}.$$  

Jahangir graph $J_{n,m}$ [11], for $m \geq 3$, is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle $C_{nm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{nm}$ at distance $n$ to each other on $C_{nm}$.

For labeling $J_{3,m}$ ($m \geq 3$), let $v_{3m+1}$ be the label of the center vertex and $v_1, v_2, \cdots, v_{3m}$ be the label of the vertices that are incident clockwise on cycle $C_{3m}$ so that $\text{deg}(v_1) = 3$. 

With regard to $t$-pebbling number of graphs, we have found the following theorems:

**Theorem 1.** ([9]). Let $K_n$ be the complete graph on $n$ vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.

**Theorem 2.** ([2]). Let $K_1 = \{v\}$. Let $C_{n-1} = (u_1, u_2, \ldots, u_{n-1})$ be a cycle of length $n - 1$. Then the $t$-pebbling number of the wheel graph $W_n$ is $f_t(W_n) = 4t + n - 4$ for $n \geq 5$.

**Theorem 3.**

$$f_t(G) = \begin{cases} 2t + n - 2, & \text{if } 2t \leq n - s_1 \\ 4t + s_1 - 2, & \text{if } 2t \geq n - s_1 \end{cases}$$

**Theorem 4.** ([9]). Let $K_{1,n}$ be an $n$-star where $n > 1$. Then $f_t(K_{1,n}) = 4t + n - 2$.

**Theorem 5.** ([9]). Let $C_n$ denote a simple cycle with $n$ vertices, where $n \geq 3$. Then $f_t(C_{2k}) = t2^k$ and $f_t(C_{2k+1}) = \frac{2^{k+2} - (-1)^{k+2}}{3} + (t - 1)2^k$.

**Theorem 6.** ([9]). Let $P_n$ be a path on $n$ vertices. Then $f_t(P_n) = t(2^{n-1})$.

**Theorem 7.** ([9]). Let $Q_n$ be the $n$-cube. Then $f_t(Q_n) = t(2^n)$.

Lourdusamy et. al proved the $t$-pebbling number of Jahangir graph $J_{2,m}$ for $m \geq 3$ and $t \geq 1$ in [3, 6, 7, 8]. In the next section, we are going to present the pebbling number of Jahangir graph $J_{3,m}$ for $m \geq 3$. In section three, we find the $t$-pebbling number for $J_{3,m}$. In the last section, we give a conjecture for the $t$-pebbling number of Jahangir graph $J_{n,m}$, where $2 < n < m$.

2. The pebbling number for Jahangir graph $J_{3,m}$

Consider Jahangir graph $J_{3,m}$ ($m \geq 3$), and we can see that $J_{3,m}$ has $m$ cycles of length five, 5-cycles, that is,

$$S_1 : v_1v_2v_3v_4v_{3m+1}v_1,$$

$$S_2 : v_4v_5v_6v_7v_{3m+1}v_4,$$

$$\vdots$$
$S_m : v_{3m-2}v_{3m-1}v_{3m}v_1v_{3m+1}v_{3m-2}$. 

For $i, j = 1, 2, \ldots, m$, $i \neq j$, two cycles $S_i$ and $S_j$ are adjacent if there exists a common edge $(v_kv_{3m+1})$ between them. The vertex $v_k$ is called a common vertex for $S_i$ and $S_j$.

**Theorem 1.** For Jahangir graph $J_{3,3}$, $f(J_{3,3}) = 17$.

**Proof.** Clearly, by Fact 2, $f(J_{3,3}) \geq \max\{16, 10\} = 16$. Let $\phi(v_6) = 15$, $\phi(v_9) = 1$ and for all $i \neq 6, 9$, $\phi(v_i) = 0$. If $v_2$ is a target vertex, then pebbling $\phi$ is $v_2$-unsolvable. Thus $f(J_{3,3}) \geq 17$.

Consider the distribution of 17 pebbles on the vertices of $J_{3,3}$.

**Case 1:** Let $v_{10}$ be the target vertex. Without loss of generality, let $\phi(V(S_1)) \geq 5$, since $J_{3,3}$ has 17 pebbles and three 5-cycles. Hence, we can move one pebble to $v_{10}$ by Theorem 5.

**Case 2:** Let $v_1$ be the target vertex. If $\phi(V(S_1)) \geq 5$ or $\phi(V(S_3)) \geq 5$, then we can easily move one pebble to $v_1$. Assume $\phi(v_2) + \phi(v_3) \leq 3$ and $\phi(v_8) + \phi(v_9) \leq 3$ (otherwise one pebble can be moved to $v_1$). Obviously, $\phi(V(S_2)) \geq 11$, and by Theorem 5, $f_2(C_5) = 9$. Therefore by moving two pebbles to $v_{10}$, we can move one pebble to $v_1$.

**Case 3:** Let $v_2$ be the target vertex. If $\phi(V(S_1)) \geq 5$, then we can easily move one pebble to $v_2$. Since, $\phi(V(J_{3,3})$--
\{v_2, v_3\} \geq 16, either \(\phi(V(S_2)) \geq 8\) or \(\phi(V(S_3)) \geq 8\).

**Case 3.1:** Let \(\phi(V(S_3)) \geq 8\).

If \(\phi(V(S_3)) \geq 9\), then by Theorem 5 we can move one pebble to \(v_2\). Let \(\phi(V(S_3)) = 8\). Clearly, \(\phi(V(S_2)) \geq 5\). Thus we can move one pebble to \(v_7\) or \(v_{10}\) from the pebbles on \(V(S_2)\). Now, we have at least 9 pebbles on \(V(S_3)\) and hence we can move one pebble to \(v_2\) through \(v_1\) by Theorem 5.

**Case 3.2:** Let \(\phi(V(S_2)) \geq 8\).

Let \(\phi(V(S_2)) \geq 9\). Clearly we can move one pebble to \(v_2\) if \(\phi(v_1) = 1\) or \(\phi(v_3) = 1\) or \(\phi(v_8) + \phi(v_9) \geq 4\). So, we assume \(\phi(v_1) = 0\), \(\phi(v_3) = 0\) and \(\phi(v_8) + \phi(v_9) \leq 3\) such that we cannot move one pebble to \(v_1\) from the pebbles on the vertices \(v_8\) and \(v_9\). Now, we have \(\phi(V(S_2)) \geq 14\). We assume \(\phi(v_4) = 0\) and \(\phi(v_{10}) = 0\) (otherwise one pebble could be moved to \(v_2\) by Theorem 5).

**Case 3.2.1:** Let \(\phi(v_8) = 2\) or \(3\) and \(\phi(v_9) = 0\).

If \(\phi(v_7) \geq 1\), then we move one pebble to \(v_7\) from \(v_8\) and then we move one pebble to \(v_{10}\) from \(v_7\). Now \(V(S_2) - \{v_{10}\}\) contains at least 13 pebbles and hence we can easily move one pebble to \(v_2\) through \(v_{10}\) by Theorem 5. Assume \(\phi(v_7) = 0\). Let \(\phi(v_8) = 2\). If \(\phi(v_6)\) \(\geq 2\), then we move one pebble \(v_{10}\) through \(v_7\) from \(v_6\) and \(v_8\) and hence we can easily move one pebble to \(v_2\) through \(v_{10}\) by Theorem 5. If \(\phi(v_6) \leq 1\), then we can easily move one pebble to \(v_2\) since \(\phi(v_5) \geq 8\) and \(d(v_2, v_5) = 3\). Let \(\phi(v_8) = 3\). If \(\phi(v_6) \geq 4\), then we move one pebble \(v_8\) from \(v_6\) and then we move one pebble to \(v_1\). Now \(V(S_2)\) contains at least 10 pebbles and hence we can move one more pebble to \(v_1\) through \(v_{10}\) (by Theorem 5) and then one pebble can be easily moved to \(v_2\). If \(\phi(v_6) \leq 3\), then we can easily move one pebble to \(v_2\) since \(\phi(v_5) \geq 8\) and \(d(v_2, v_5) = 3\).

**Case 3.2.2:** Let \(\phi(v_8) = \phi(v_9) = 1\).

If \(\phi(v_6) + \phi(v_7) \geq 4\), then we can move one pebble to \(v_1\) using the pebbles on \(v_8\) and \(v_9\). Then \(\phi(V(S_2)) - 4 \geq 10\) and hence we can move another one pebble to \(v_1\) through \(v_{10}\) and then one pebble can be easily moved to \(v_2\). Suppose, \(\phi(v_6) + \phi(v_7) \leq 3\), then \(\phi(v_5) \geq 8\). Thus we can easily move one pebble to \(v_2\) since \(d(v_2, v_5) = 3\).

**Case 3.2.3:** Let \(\phi(v_8) + \phi(v_9) \leq 1\).

Since \(\phi(v_8) + \phi(v_9) \leq 1\), we have \(\phi(V(S_2)) \geq 16\). If \(\phi(v_5) \geq 2\), then we move
A. Lourdusamy and T. Mathivanan

one pebble to $v_4$. Then the number of remaining pebbles on the vertices of $S_2$ is at least 14 and hence we can move three more pebbles to $v_4$ by Theorem 5. Thus we can easily move one pebble to $v_2$ from $v_4$. Assume $\phi(v_5) \leq 1$. In a similar way, we may assume $\phi(v_7) \leq 1$. This implies that $\phi(v_6) \geq 14$ and clearly we can move one pebble to $v_2$ from $v_4$. Assume $\phi(v_5) = 0$ and $\phi(v_7) = 0$. Then $\phi(v_6) = 16$ and hence one pebble can be moved to $v_2$ since $d(v_2, v_6) = 4$.

Therefore 17 pebbles are sufficient to pebble the vertices of $J_{3,3}$ and hence $f(J_{3,3}) = 17$. 

Theorem 2. For Jahangir graph $J_{3,4}$, $f(J_{3,4}) = 21$.

Proof. Let $\phi(v_8) = 15$, $\phi(v_5) = 3$, $\phi(v_9) = \phi(v_{12}) = 1$ and $\phi(v_i) = 0$ for all $i \neq 5, 8, 9, 12$. Then we cannot move one pebble to $v_2$. Thus $f(J_{3,4}) \geq 21$.

Figure 2. Jahangir graph $J_{3,4}$

Consider the distribution of 21 pebbles on the vertices of $J_{3,4}$.

Case 1: Let $v_{13}$ be the target vertex.
Without loss of generality, let $\phi(V(S_1)) \geq 5$, since $J_{3,4}$ has 21 pebbles and four 5-cycles. Hence, we can move one pebble to $v_{13}$ by Theorem 5.

Case 2: Let $v_1$ be the target vertex.
If $\phi(V(S_1)) \geq 5$ or $\phi(V(S_4)) \geq 5$, then we can easily move one pebble to $v_1$. Assume $\phi(v_2) + \phi(v_3) \leq 3$ and $\phi(v_{11}) + \phi(v_{12}) \leq 3$ (otherwise one pebble can be moved to $v_1$). Thus, $\phi(V(S_2)) + \phi(V(S_3)) \geq 15$. If both
\[\phi(V(S_2)) \geq 5 \text{ and } \phi(V(S_3)) \geq 5, \text{ then we can easily move one pebble to } v_1. \] Without loss of generality, let \(\phi(V(S_3)) \leq 4. \) So, \(\phi(V(S_2)) \geq 11\) and hence we can move one pebble to \(v_1\) by moving two pebbles to \(v_{13}\), since \(f_2(C_5) = 9\) by Theorem 5.

**Case 3:** Let \(v_2\) be the target vertex.
If \(\phi(V(S_1)) \geq 5\), then clearly we can move one pebble to \(v_2\). So, we have \(\phi(V(S_2)) + \phi(V(S_3)) + \phi(V(S_4)) \geq 20\) and hence \(\phi(V(S_i)) \geq 7\) for some \(i = 2, 3, 4\).

**Case 3.1:** Let \(\phi(V(S_2)) \geq 7\).
Let \(\phi(V(S_2)) \geq 9.\) If \(\phi(v_1) = 1\) or \(\phi(v_3) = 1\) or \(\phi(v_{11}) + \phi(v_{12}) \geq 4\), then clearly we can move one pebble to \(v_2\). Assume \(\phi(v_1) = 0, \phi(v_3) = 0\) and \(\phi(v_{11}) + \phi(v_{12}) \leq 3\) such that we cannot move one pebble to \(v_1\). We have \(\phi(V(S_2)) + \phi(V(S_3)) \geq 18\).

**Case 3.1.1:** Let \(\phi(v_{11}) = 0\) and \(\phi(v_{12}) = 2\) or \(\phi(v_{11}) = 0\) and \(\phi(v_{12}) = 3\).
If \(\phi(v_{10}) = 1\) or \(\phi(v_9) \geq 2\), then we move one pebble to \(v_{13}\). If \(\phi(V(S_3)) - 2 \geq 5\), then we can move one more pebble to \(v_{13}\) and hence we are done since \(\phi(V(S_2)) \geq 9\). Assume that \(\phi(V(S_3)) - 2 \leq 4\). Thus \(\phi(V(S_2)) \geq 12\).
If \(\phi(v_8) \geq 2\), then we move one pebble to \(v_7\). Now, \(V(S_2)\) contains at least 13 pebbles and hence we can move three pebbles to \(v_{13}\) and so we can move one pebble to \(v_2\). Let \(\phi(v_8) \leq 1.\) Clearly, \(\phi(V(S_2)) \geq 13\) and hence we can move one pebble to \(v_2\) through \(v_{13}\). Let \(\phi(v_{10}) = 0, \phi(v_9) \leq 1\) and \(\phi(v_8) \leq 3.\) Thus, \(\phi(V(S_2)) \geq 15.\) If \(\phi(v_{13}) = 1\) or \(\phi(v_4) = 1\) or \(\phi(v_5) \geq 2\) or \(\phi(v_7) \geq 2\), then we can move three additional pebbles to either \(v_4\) or \(v_{13}\), since \(\phi(V(S_2)) - 2 \geq 13\) and hence we can easily move one pebble to \(v_2\). So, we assume that \(\phi(v_{13}) = 0, \phi(v_4) = 0, \phi(v_5) \leq 1\) and \(\phi(v_7) \leq 1\) and thus \(\phi(v_6) \geq 13.\) Let \(\phi(v_7) = 1.\) If \(\phi(V(S_3)) = 4,\) then we can move one pebble to \(v_7\) and then we move six pebbles to \(v_7\) from \(v_6\) and hence we can move one pebble to \(v_2,\) since \(d(v_2, v_7) = 3.\) If not, then we can move seven pebbles to \(v_7\) from \(v_6\) and hence we are done. Thus we assume \(\phi(v_7) = 0.\) For the same reason we assume \(\phi(v_5) = 0\) and thus \(\phi(v_6) \geq 15.\) If \(\phi(V(S_3)) \geq 3,\) then we can move one pebble to \(v_7\) and then we move seven pebbles to \(v_7\) from \(v_6\) and hence we can move one pebble to \(v_2,\) since \(d(v_2, v_7) = 3.\) If not, then we can move eight pebbles to \(v_7\) from \(v_6\) and hence we are done.

**Case 3.1.2:** Let \(\phi(v_{11}) = 1\) and \(\phi(v_{12}) = 1.\)
Clearly, we can move one pebble to \(v_1\) using the pebbles on \(v_{11}\) and \(v_{12}\)
if \( \phi(V(S_3)) \geq 7 \) and then we move one more pebble to \( v_1 \) from \( \phi(V(S_2)) \) and hence we are done. Assume \( \phi(V(S_3)) \leq 6 \) and so \( \phi(V(S_2)) \geq 13 \). If \( \phi(V(S_3)) \geq 5 \), then also we can easily move one pebble to \( v_2 \). Assume \( \phi(V(S_3)) \leq 4 \) and thus \( \phi(V(S_2)) \geq 15 \). We do the same thing as we did above when \( \phi(V(S_2)) \geq 15 \) to put one pebble at \( v_2 \). Next, we assume \( \phi(V(S_2)) = 7 \) or \( 8 \). If \( \phi(V(S_3)) \geq 5 \) and \( \phi(V(S_4)) \geq 5 \), then we can easily move one pebble to \( v_2 \). Assume that \( \phi(V(S_3)) \leq 4 \) and so \( \phi(V(S_4)) \geq 8 \). Either we can move one more pebble to \( v_{10} \) from \( \phi(V(S_3)) \) or we should have \( \phi(V(S_4)) \geq 9 \) and hence we can easily move one pebble to \( v_2 \). Assume that \( \phi(V(S_4)) \leq 4 \) and so \( \phi(V(S_3)) \geq 8 \). If \( \phi(v_{11}) + \phi(v_{12}) \geq 4 \) or \( \phi(v_3) = 1 \), then we can move one pebble to \( v_2 \) either through \( v_1 \) or \( v_3 \). So we assume that \( \phi(v_{11}) + \phi(v_{12}) \leq 3 \) and \( \phi(v_3) = 0 \) and hence we can move 'four pebbles to \( v_{13} \)' or 'one pebble to \( v_1 \) and two pebbles to \( v_{13} \)', like we did above. Thus we can easily move one pebble to \( v_2 \).

**Case 3.2:** Let \( \phi(V(S_4)) \geq 7 \).
Either \( \phi(V(S_4)) \geq 9 \) or we can add one or two pebbles (if \( \phi(V(S_4)) = 8 \) or \( 7 \)) to \( V(S_4) \) from \( V(S_2) \) and \( V(S_3) \). Thus we can easily move one pebble to \( v_2 \).

**Case 3.3:** Let \( \phi(V(S_3)) \geq 7 \).
Let \( \phi(V(S_3)) \geq 9 \). Clearly, we can move one pebble to \( v_2 \) if \( \phi(V(S_4)) \geq 5 \) or \( \phi(v_1) = 1 \) or \( \phi(v_{11}) + \phi(v_{12}) \geq 4 \). So we assume that \( \phi(v_1) = 0 \) and \( \phi(v_{11}) + \phi(v_{12}) \leq 3 \).

**Case 3.3.1:** Let \( \phi(V(S_2)) \geq 7 \).
Clearly, we can move one pebble to \( v_2 \) by Case 3.1.

**Case 3.3.2:** Let \( \phi(V(S_2)) = 5 \) or \( 6 \).
Clearly we can move one pebble to \( v_2 \) if \( \phi(v_3) = 1 \). So, we assume \( \phi(v_3) = 0 \). If \( \phi(v_{11}) \geq 2 \), then we move one pebble to \( v_{10} \) and hence \( V(S_3) \) contains 13 pebbles. Thus we can move four pebbles to \( v_{13} \) from \( V(S_2) \) and \( V(S_3) \) and hence we can move one pebble to \( v_2 \). Let \( \phi(v_{11}) \leq 1 \). Thus, \( \phi(S_3) \geq 13 \) and hence we can move one pebble to \( v_2 \) through \( v_{13} \).

**Case 3.3.3:** Let \( \phi(V(S_2)) \leq 4 \).
We have \( \phi(V(S_3)) \geq 13 \). Hence, we can move 'four pebbles to \( v_{13} \)' or 'one pebble to \( v_1 \) and two pebbles to \( v_{13} \)', by considering the cases \( \phi(v_{11}) = 2 \) or \( 3 \) or \( \phi(v_{11}) = 1 \) and \( \phi(v_{12}) = 1 \) or \( \phi(v_{12}) \leq 1 \). Otherwise, \( \phi(V(S_3)) \geq 17 \) and hence we can easily move one pebble to \( v_2 \) by Theorem
The t-pebbling number of Jahangir graph $J_{3,m}$

5.

For the case $\phi(V(S_3)) = 7$ or 8, one could see that we can easily always move a pebble to $v_2$.

Thus we can always move one pebble to $v_2$ using 21 pebbles on the vertices of $J_{3,4}$. So, $f(J_{3,4}) = 21$. □

**Theorem 3.** For Jahangir graph $J_{3,m}$ $(m \geq 5)$, $f(J_{3,m}) = 3m + 10$.

**Proof.** Consider the following distribution: $\phi(v_8) = 15$, $\phi(v_9) = \phi(v_{3m}) = \phi(v_{3m-1}) = 1$, $\phi(v_4) = 3$ and $\phi(v_i) = 3$ for all $i \in \{12 + 3k\}$ ($0 \leq k \leq m - 5$). Then we cannot move one pebble to $v_2$. Since this distribution contains $15 + 1 + 1 + 3 + (m - 4)3 = 3m + 9$ pebbles, $f(J_{3,m}) \geq 3m + 10$ for $m \geq 5$.

Now, we consider the distribution of $3m + 10$ pebbles on the vertices of $J_{3,m}$ where $m \geq 5$.

**Case 1:** Let $v_{3m+1}$ be the target vertex.

If any one of the 5-cycle contains five or more pebbles, then we can easily move one pebble to $v_{3m+1}$. Consider every 5-cycle contains at most four pebbles only. Since we have placed $3m + 10$ pebbles on the vertices of $J_{3,m}$, ten 5-cycles must contain exactly four pebbles. Without loss of generality, let $\phi(V(S_1)) = 4$. If one of adjacent cycle also has four pebbles or the adjacent vertex of a common vertex from the adjacent cycle contains more than two pebbles, then we can move one pebble to $v_{3m+1}$ through the common vertex $v_1$ or $v_4$. If both the adjacent vertices of a common vertex have more than one pebble each, then also we can move one pebble to $v_{3m+1}$. Otherwise, the graph $J_{3,m}$ must contain at most $3m + 1$ pebbles - which is a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$.

**Case 2:** Let $v_1$ be the target vertex.

If $\phi(V(S_i)) \geq 5$ or $\phi(V(S_m)) \geq 5$, then we can easily move one pebble to $v_1$. Also if $\phi(v_2 + v_3) \geq 4$ or $\phi(v_{3m} + v_{3m-1}) \geq 4$, then we can move one pebble to $v_1$. So, we assume $\phi(v_2 + v_3) \leq 3$ and $\phi(v_{3m-1} + v_{3m}) \leq 3$. If $\phi(V(S_i)) \geq 5$ and $\phi(V(S_j)) \geq 5$ for some $i \neq 1, m$, $j \neq 1, m$, then we can move two pebbles to $v_{3m+1}$ and hence one pebble is moved to $v_1$. Assume $\phi(V(S_i)) \geq 5$ and all other cycles contain at most four pebbles each except $S_1$ and $S_m$. Suppose we cannot move one more pebble to $v_{3m+1}$ or we cannot move one pebble to $v_1$, then the graph $J_{3,m}$ contains at most $3m + 2$ pebbles - a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$. Next, we assume that every cycle contains at most...
four pebbles only. If we have two adjacent cycles with four pebbles each on them, then we can move two pebbles to \( v_{3m+1} \) and hence we move one pebble to \( v_1 \). Thus we assume only one adjacent copy has four pebbles each on them. We move one pebble to \( v_{3m+1} \) from that adjacent cycle. Suppose if we cannot move one more pebble to \( v_{3m+1} \) or if we cannot move one pebble to \( v_1 \), then the graph has at most \( 3m + 2 \) pebbles - a contradiction to the total number of pebbles placed on the vertices of \( J_{3,m} \). Suppose if there is no such adjacent cycles, then we can move two pebbles to \( v_{3m+1} \), since we have \( 3m + 10 \) pebbles and ten cycles have exactly four pebbles. If we cannot move one pebble to \( v_{3m+1} \) or if we cannot move one pebble to \( v_1 \), then the graph \( J_{3,m} \) has at most \( 3m \) pebbles - a contradiction to the total number of pebbles placed on the vertices of \( J_{3,m} \).

Case 3: Let \( v_2 \) be the target vertex.

If \( \phi(V(S_1)) \geq 5 \), then clearly we can move one pebble to \( v_2 \). Suppose four cycles have five pebbles each on them. Then we can move four pebbles to \( v_{3m+1} \) pebbles and hence one pebble is moved to \( v_2 \). Let three cycles only have more than four pebbles. So we can move three pebbles to \( v_{3m+1} \). If we cannot move one more pebble to \( v_{3m+1} \) or if we cannot move one pebble to \( v_2 \), then the graph \( J_{3,m} \) has at most \( 3m + 8 \) pebbles - a contradiction. Let two cycles have more than four pebbles. Suppose if we cannot move one pebble to \( v_2 \), then the graph has at most \( 3m + 6 \) pebbles - a contradiction. Let only one cycle has more than four pebbles. Suppose if we cannot move one pebble to \( v_2 \), then the graph has at most \( 3m + 9 \) pebbles - a contradiction. Assume every cycle has at most four pebbles only. Suppose if we cannot move one pebble to \( v_2 \), then the graph has at most \( 3m + 5 \) pebbles - a contradiction.

Thus we can always move one pebble to \( v_2 \) using \( 3m + 10 \) pebbles on the vertices of \( J_{3,m} \). So, \( f(J_{3,m}) = 3m + 10 \).

3. The \( t \)-pebbling number of Jahangir graph \( J_{3,m} \)

Theorem 1. For Jahangir graph \( J_{3,3} \), \( f_t(J_{3,3}) = 16t + 1 \).

Proof. Let \( \phi(v_6) = 16(t - 1) + 15 \), \( \phi(v_9) = 1 \) and \( \phi(v_i) = 0 \) for all \( i \neq 6, 9 \). Then we cannot move \( t \) pebbles to \( v_2 \). Thus \( f_t(J_{3,3}) > 16t \).

Now, consider the distribution of the \( 16t + 1 \) pebbles on the vertices of \( J_{3,3} \). Clearly the result is true for \( t = 1 \). Assume the result is true for \( 2 \leq t' < t \). Clearly, the graph \( J_{3,3} \) has at least 33 pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles.
Then the remaining number of pebbles on the vertices of $J_{3,3}$ is at least $16(t-1) + 1$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,3}) \leq 16t + 1$. □

**Theorem 2.** For Jahangir graph $J_{3,4}$, $f_t(J_{3,4}) = 16t + 5$.

**Proof.** Let $\phi(v_8) = 16(t-1) + 15$, $\phi(v_5) = 3$, $\phi(v_9) = \phi(v_{12}) = 1$ and $\phi(v_i) = 0$ for all $i \neq 5, 8, 9, 12$. Then we cannot move one pebble to $v_2$. Thus $f_t(J_{3,4}) > 16t + 4$.

Now, consider the distribution of the $16t + 5$ pebbles on the vertices of $J_{3,4}$. Clearly the result is true for $t = 1$. Assume the result is true for $2 \leq t' < t$. Clearly, the graph $J_{3,4}$ has at least 37 pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles. Then the remaining number of pebbles on the vertices of $J_{3,4}$ is at least $16(t-1) + 5$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,4}) \leq 16t + 5$. □

**Theorem 3.** For Jahangir graph $J_{3,m}$ ($m \geq 5$), $f_t(J_{3,m}) = 16t + 3m - 6$.

**Proof.** Consider the following distribution: $\phi(v_8) = 16(t-1) + 15$, $\phi(v_9) = \phi(v_{3m}) = \phi(v_{3m-1}) = 1$, $\phi(v_4) = 3$ and $\phi(v_i) = 3$ for all $i \in \{12 + 3k\} (0 \leq k \leq m - 5)$. Then we cannot move one pebble to $v_2$. Since this distribution contains $16(t-1) + 3m + 9$ pebbles, $f_t(J_{3,m}) \geq 16t + 3m - 6$ for $m \geq 5$.

Now, consider the distribution of the $16t+3m-6$ pebbles on the vertices of $J_{3,m}$. Clearly the result is true for $t = 1$. Assume the result is true for $2 \leq t' < t$. Clearly, the graph $J_{3,m}$ has at least $3m + 24$ pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles. Then the remaining number of pebbles on the vertices of $J_{3,m}$ is at least $16(t-1) + 3m - 6$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,m}) \leq 16t + 3m - 6$. □

4. An upper bound for the $t$-pebbling number of Jahangir graph $J_{n,m}$

Here, we present the known results about the pebbling number of Jahangir graphs from [6, 7, 8]. The pebbling number of Jahangir graph $J_{2,m}$ ($m \geq 3$) is as follows:

**Theorem 1.** [6] For Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$. 

Theorem 2. [6] For Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$.

Theorem 3. [6] For Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.


Theorem 5. [6] For Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.

Theorem 6. [7] For Jahangir graph $J_{2,m}$ where $m \geq 8$, $f(J_{2,m}) = 2m + 10$.

The $t$-pebbling number of Jahangir graph $J_{2,m}$ ($m \geq 3$) is as follows:

Theorem 7. [8] For Jahangir graph $J_{2,3}$, $f_t(J_{2,3}) = 8t$.

Theorem 8. [8] For Jahangir graph $J_{2,4}$, $f_t(J_{2,4}) = 16t$.

Theorem 9. [8] For Jahangir graph $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.

Theorem 10. [8] For Jahangir graph $J_{2,m}$, $f_t(J_{2,m}) = 16(t - 1) + f(J_{2,m})$ where $m \geq 6$.

From the above results and the results from this paper, we can conclude that $f_t(J_{n,m}) \geq t(2^k)$, where $k = 2^{2\left\lfloor \frac{n}{2} \right\rfloor + 2}$ is the diameter of $J_{n,m}$ for $3 \leq n < m$ (for $n = 2$, we take $m \geq 4$).

After seeing the behaviour of Jahangir graph $J_{n,m}$, we give the following conjecture for the $t$-pebbling number of $J_{n,m}$.

Conjecture 1. For Jahangir graph $J_{n,m}$ ($3 \leq n < m$),

$$f_t(J_{n,m}) \leq \begin{cases} 
  t(2^k) + (m - 2)(2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1) & \text{if } n \text{ is even} \\
  t(2^k) + (m - 3)(2^{\left\lfloor \frac{n}{2} \right\rfloor + 1} - 1) + n & \text{if } n \text{ is odd}
\end{cases}$$

where $k = 2^{2\left\lfloor \frac{n}{2} \right\rfloor + 2}$ is the diameter of $J_{n,m}$.

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