ON CONMUTATIVE LEFT-NILALGEBRAS OF INDEX 4 *

JUAN C. GUTIERREZ FERNANDEZ
UNIVERSIDADE DE SAO PAULO, BRASIL

Received : Diciembre 2007. Accepted : March 2008

Abstract

We first present a solution to a conjecture of (Correa, Hentzel, Labra, 2002) in the positive. We show that if $A$ is a commutative nonassociative algebra over a field of characteristic $\neq 2, 3$, satisfying the identity $x(x(xx)) = 0$, then $L_{a^{t_1}}L_{a^{t_2}}\cdots L_{a^{t_s}} \equiv 0$ if $t_1 + t_2 + \cdots + t_s \geq 10$, where $a \in A$.

Keywords : solvable; commutative; nilalgebra;

Mathematics Subject Classification : 17A05, 17A30

*Partially Supported by FAPESP 05/01790-9, Brasil
1. Introduction

Throughout this paper the term *algebra* is understood to be a commutative not necessarily associative algebra. We will use the notations and terminology of (Fernandez, 2004). Let $A$ be an (commutative nonassociative) algebra over a field $F$. We define inductively the following powers, $A^1 = A$ and $A^s = \sum_{i+j=s} A^i A^j$ for all positive integers $s \geq 2$. We shall say that $A$ is *nilpotent* if there is a positive integer $s$ such that $A^s = (0)$. The least such number is called the index of nilpotency of the algebra $A$. The algebra $A$ is called *nilalgebra* if given $a \in A$ we have that $\text{alg}(a)$, the subalgebra of $A$ generated by $a$, is nilpotent. The *(principal) powers* of an element $a$ in $A$ are defined recursively by $a^1 = a$ and $a^{i+1} = aa^i$ for all integers $i \geq 1$. The algebra $A$ is called *left-nilalgebra* if for every $a$ in $A$ there exists an integer $k = k(a)$ such that $a^k = 0$. The smallest positive integer $k$ which this property is the *index*. Obviously, every nilalgebra is left-nilalgebra. For any element $a$ in $A$, the linear mapping $L_a$ of $A$ defined by $x \rightarrow ax$ is called *multiplication operator* of $A$. An *Engel algebra* is an algebra in which every multiplication operator is nilpotent in the sense that for every $a \in A$ there exists a positive integer $j$ such that $L_a^j = 0$.

An important question is that of the existence of simple nilalgebras in the class of finite-dimensional algebras. In (Fernandez, 2004) we proved that every nilalgebra $A$ of dimension $\leq 6$ over a field of characteristic $\neq 2, 3, 5$ is solvable and hence $A^2 A$. For power-associative nilalgebras of dimension $\leq 8$ over a field of characteristic $\neq 2, 3, 5$, we have shown in (Fernandez, Suazo, 2005) that they are solvable, and hence there is no simple algebra in this subclass. See also (Elgueta, Suazo, 2004; Fernandez, 2004) for power-associative nilalgebras of dimension $\leq 7$.

We show now the process of linearization of identities, which is an important tool in the theory of varieties of algebras. See (Gerstenhaber, 1960; Osborn, 1972; Zhevlakov, 1982) for more information. Let $P$ be the free commutative nonassociative polynomial ring in two generators $x$ and $y$ over a field $F$. For every $\alpha_1, \ldots, \alpha_r \in P$, the *operator linearization* $\delta[\alpha_1, \ldots, \alpha_r]$ can be defined as follows: if $p(x, y)$ is a monomial in $P$, then $\delta[\alpha_1, \ldots, \alpha_r]p(x, y)$ is obtained by making all the possible replacements of $r$ of the $k$ identical arguments $x$ by $\alpha_1, \ldots, \alpha_r$ and summing the resulting terms if $x$–degree of $p(x, y)$ is $\geq r$, and is equal to zero in other cases. Some examples of this operator are

$$ [y][y](x^2(xy)) = 2(xy)^2 + x^2y^2 $$

$$ \delta[x^2, y](x^2) = 2x^2y, \quad \delta[y, xy^2, x](x^2) = 0. $$

For simplicity, $\delta[\alpha : r]$ will denote $\delta[\alpha_1, \ldots, \alpha_r]$, where $\alpha_1 = \cdots = \alpha_r = \alpha$. We observe that if $p(x)$ is a polynomial in $P$, then $p(x + y) = p(x) + \sum_{j=1}^{\infty} \delta[y : j]p(x)$, where $\delta[y : j]p(x)$ is the sum of all the terms of $p(x + y)$ which have degree $j$ with respect to $y$.

**Lemma 1.** (Zhevlakov, 1982) Let $p(x, y)$ be a commutative nonassociative polynomial
of $x$-degree $\leq n$. If $F$ is a field of characteristic either zero or $\geq n$, and the $F$-algebra $A$ satisfies the identity $p(x,y)$, then $A$ satisfies all linearizations of $p(x,y)$.

2. Left-nilalgebras of index 4

Throughout this section $F$ is a field of characteristic different from 2 or 3 and all the algebras are over $F$. We will study left-nilalgebras of index $\leq 4$, that is the variety $V$ of algebras over the field $F$ satisfying the identity

$$x^4 = 0.$$  

Let $A$ be an algebra in $V$. For simplicity, we will denote by $L$ and $U$ the multiplication operators, $L_x$ and $L_{x^2}$ respectively, where $x$ is an element in $A$. The following known result is a basic tool in our investigation. See (Correa, Hentzel, Labra, 2002; Elduque, Labra, 2007).

**Lemma 2.** Let $A$ be a commutative left-nilalgebra of index 4. Then $A$ satisfies the identities

$$xx = -x(xx), \quad x^3 = x(x(xx)),$$

and $p(x) = 0$, for every monomial $p(x)$ with $x$-degree $\geq 7$. Furthermore, we have

$$[lcl]L_x = -LU - 2L^3,$$  

$$L_{x^2} = -U - 2UL^2 - 2LUL + 4L^4,$$  

$$L_{xx} = -LU - 2LUL - 2LUL - 4L^3U - 12L^5,$$  

$$L_{x(xx)} = 2LU + 4LUL + 4L^4U + 8L^6,$$

and also

<table>
<thead>
<tr>
<th>$UL$</th>
<th>$LU^2$</th>
<th>$UL^3$</th>
<th>$LUL^2$</th>
<th>$L^2UL$</th>
<th>$L^3U$</th>
<th>$L^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U^2L$</td>
<td>0</td>
<td>$-1$</td>
<td>2</td>
<td>0</td>
<td>$-2$</td>
<td>$-8$</td>
</tr>
</tbody>
</table>

and two identities of $x$-degree 6 which may be written as

<table>
<thead>
<tr>
<th>$UL^2U$</th>
<th>$(UL)^2$</th>
<th>$L^2U^2$</th>
<th>$UL^4$</th>
<th>$LU^3$</th>
<th>$L^2UL$</th>
<th>$L^3UL$</th>
<th>$L^4U$</th>
<th>$L^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U^3$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>2</td>
<td>$-8$</td>
<td>$-8$</td>
<td>0</td>
<td>$-4$</td>
<td>8</td>
</tr>
<tr>
<td>$(UL)^2$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
<td>$-4$</td>
<td>$-2$</td>
<td>2</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

We note that, for example, Table i means that $U^2L = -LU^2 + 2UL^3 - 2L^3U - 8L^5$. From the identities (2.3-2.6) we get that for any $a \in A$ the associative algebra $A_a$ generated by all $L_c$ with $c \in \text{alg}(a)$ is in fact generated by $L_a$ and $L_a^2$. Furthermore, every algebra in $V$ is a nilalgebra of index $\leq 7$. 

We now pass to study homogeneous identities in $A$ with $x$-degree $\geq 7$ and $y$-degree 1. From the relation $0 = \delta [y, x, x, x] (x^4) = 2y(x(x^3x^3)) + 4y(x^3(xx^3)) + 2x(y(x^3x^3)) + 4x^3(y(x^3x^3)) + 4x(x^3(yx^3)) + 4x^3(xyx^3)) + 4x^3(x^3(xy)) = 2[(x^3)^2y] + 4x(x^3(xy)) + 4x^3(xy) + 4x^3(x^3(xy)) = 2[LL_x x_3 + 2L_xx_3 + 2L_xL_xL_x callee we get a new identity as

\[
L^3U^2 = -2L^3UL - L^4UL - 5L^5U - 20L^7,
\]

since we can use the reductions (2.3–2.6) and replace the occurrences of $(UL)^2$. Multiplying the identity of Table i by $U$ from the left, replacing first the occurrences of $U^3$ and next using reductions from Table i, Table ii and above identity we get a new identity as follows:

\[
0 = U^3L + ULU^2 - 2U^2L^3 + 2ULU^3 + 8UL^5 = [2UL^2UL - 2(UL)^2L + 2L^2U^2L - 8UL^5
- 8UL^4 - 4L^3UL^2 + 8L^4UL + 40L^7] + ULU^2 - 2U^2L^3 + 2UL^2U + 8UL^5
\]

\[
= -2UL^2UL + [2ULU^2 + 2L(UL)^2] + [4L^2UL^2 + 2L^4UL + 10L^5U + 40L^7]
+ [8UL^4 + 4L^2UL^3 - 8L^5U + 48L^7] + 2L^2U^2L - 8UL^5 - 8UL^4
- 4L^3UL^2 + 8L^4UL + 40L^7 + ULU^2 = [-2L^2U^2L + 4LUL^4 - 4L^4UL - 16L^7]
\]

that is,

\[
ULU^2 = 2\left(U L^2UL - UL^3U - LUL^2U - L^2ULU + 2UL^5 - 2LUL^4 - 2L^2UL^3 - 3L^4UL - L^5U - 16L^7\right).
\]

Next, we can reduce the relation $0 = \delta [y, x, x, xx] (x^4)$ using the above identities. This yields

\[
UL^5 = -LUL^4 + \frac{1}{2}L^2UL^3 + \frac{3}{4}L^4UL + \frac{3}{4}L^5U + 8L^7.
\]

Now combining (2.8) and (2.9) we obtain $ULU^2 = 2ULUL - 2UL^3U - 2LUL^2U
- 2L^2UL - 8UL^4 - 2L^2UL^3 - 3L^4UL + L^5U$. Thus, we have three identities of $x$-degree 7 and $y$-degree 1 which may be written as multiplication identities:

Table iii. Multiplication identities of degree 7.

<table>
<thead>
<tr>
<th>$UL^2UL$</th>
<th>$UL^3U$</th>
<th>$ULUL$</th>
<th>$L(UL)^2$</th>
<th>$UL^4$</th>
<th>$UL^5$</th>
<th>$L^2UL$</th>
<th>$L^4UL$</th>
<th>$L^5UL$</th>
<th>$L^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^3U$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>-5</td>
<td>-20</td>
</tr>
<tr>
<td>$UL^5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1/2</td>
<td>0</td>
<td>3/4</td>
<td>3/4</td>
</tr>
<tr>
<td>$ULU^2$</td>
<td>2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-8</td>
<td>-2</td>
<td>0</td>
<td>-3</td>
<td>1</td>
</tr>
</tbody>
</table>

In an analogous way, using successively the identities

\[
0 = \delta [y, x, x, x(x(x^2x))] (x^4), \quad 0 = \delta [y, x, x, x^2x] (x^4), \quad 0 = \delta [y, x, x^2, x(x^2x)] (x^4),
\]

multiplying the second identity of Table ii with the operator $U$ from the left and replacing the occurrences of $ULL$, and finally using $0 = \delta [y, x, x^3, x^2x] (x^4)$, we obtain the following 5 multiplication identities:
Now, relations $0 = \delta[y, x, x^3, x(x^2x^2)]x^4$, $0 = \delta[y, x, x^2, x(x(x^2x^2))]x^4$, $0 = \delta[y, x^2, x^2, (x^2x^2)]x^4$, $0 = \delta[y, x, x^2x^2, x^2x^2]x^4$, $0 = \delta[y, x^2, x^3, x^2x^2]x^4$, and multiplying the relation determined by the last row of Table iii with the operator $U$ from the left and first replacing the occurrences of $UUL$, imply the following 6 multiplication identities:

<table>
<thead>
<tr>
<th>Table iv. Multiplication identities of degree 8.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$UL^4U$</td>
</tr>
<tr>
<td>$L^3ULU$</td>
</tr>
<tr>
<td>$ULU^2$</td>
</tr>
<tr>
<td>$(UL)^2$</td>
</tr>
<tr>
<td>$UL^3UL$</td>
</tr>
<tr>
<td>$L^3UL^3$</td>
</tr>
</tbody>
</table>

The author used a simples MAPLE language program to check these identities. We now present a solution of a Conjecture of (Correa, Hentzel, Labra, 2002) in the positive. We see that for every $a \in A$, the associative algebra $A_a$, generated by the multiplication operators $L_a$ and $L_{a^2}$, is nilpotent of index $\leq 10$.

Theorem 1. Let $A$ be an algebra over a field $F$ of characteristic $\neq 2, 3$, satisfying $x^4 = 0$. Then every monomial in $P$ of $x$-degree $\geq 10$ and $y$-degree 1 is an identity in $A$. In particular, $L_a^{10} = 0$ for all $a \in A$.

Proof. First we shall prove that every monomial of $x$-degree 10 and $y$-degree 1 is an identity in $A$. Multiplying the operators in the first line of Table v with $L$ from the left and from the right, and the operators in the first line of Table iv with $U$ from the left and from the right and next using reductions from Tables i-v we see that we only need to prove that $L^2UL^4U = 0, L^8U = 0$ and $L^{10} = 0$ are multiplication identities in $A$. Now, for any $x$ in $A$ we have

$$[U]L^7UL = L(L^6UL) = -7L^8U - 48L^{10},$$
$$L^6UL^2 = (L^6UL)L = -7L^7UL - 48L^{10} = 49L^8U + 288L^{10},$$
$$L^5UL^2 = L(L^5UL^2) = -23L^8U + 288/3L^{10}.$$

Therefore

$$27L^8U + 170L^{10} = 0.$$
Now,$$
[ll]L^5UL^3 = (L^5UL^2)L = -23L^7UL - 496/3L^{10} = 161L^8U + 2816/3L^{10},$$
and hence

$$L^5UL^3 = L^2(L^3UL^3) = -3/4L^6UL^2 - 3/2L^7UL - 3/4L^8U - 8L^{10}
= -27L^8U - 152L^{10},$$

(2.11) $$141L^8U + 818L^{10} = 0.$$

Next

$$[ll]L^3UL^3U = L(L^2UL^3U) = 29/3L^8U + 422/9L^{10},$$
$$L^3UL^3U = (L^3UL^3)U = -3/4L^4UL^2U - 3/2L^5ULU - 3/4L^6UU - 8L^8U$$
$$= -3/4L^5UL^2U - 3/2L^2(L^3ULU) - 3/4L^3(L^3U^2) - 8L^{10}
= 9/4L^5UL^2 + 15/4L^7UL + 667/4L^8U + 2345/2L^{10}
= 1003L^8U + 3281/2L^{10},$$

(2.12) $$17880L^8U + 28685L^{10} = 0.$$

Combining (2.10-2.12) we obtain that $L^8U = 0$ and $L^{10} = 0$. Now, we have by Table v

that $0 = (L^2UL^3U)L = L^2(UL^3UL) = -L^2UL^4U - L^3UL^2UL - L^3UL^3U - 4L^4UL^4 - 11/2L^6UL^2 - 3L^7UL + 9/2L^8U = -L^2UL^4U - (L^3UL^2U)L - 4L(L^3UL^3) = -L^2UL^4U.$

Therefore, we have $L^2UL^4U = 0.$

In an analogous way, we can see that every monomial of $x$-degree 11 and $y$-degree 1 is an identity in $A$. This proves the theorem. \(\square\)

Now we shall investigate two subvarieties of $V$. We start in Subsection 2.1 with the class of all nilalgebras in $V$ of index $\leq 5$ and next in Subsection 2.2 we study the multiplication identities of the variety of all the nilalgebras in $V$ of index $\leq 6$.

### 2.1. The identity $x((xx)(xx)) = 0$

We will now consider the class of all algebras in $V$ satisfying the identity $x(xx) = 0$. First, linearization $\delta[y]{x(x^2)^2}$ implies

$$L_{x^2x^2} = -4LUL,$$

(2.13)

and identity $\delta[y]{x^2x^3} = 0$ forces

$$UU = -2ULL + 2LUL + 4L^4.$$

(2.14)

Next, using above identity and $\delta[y, x^2]x(x^2)^2 = 0$ we get that $0 = 4UU + 4LUU + 8LLUL = 4(UUL + LUU - 2LLL - 4L^5) = 8(-UL^3 + LULL + 2L^5 - LULL + LLUL +...}$
\[2L^5 - LLUL - 2L^5 = 8(-UL^3 + 2L^5)\]. Hence \(UL^3 = 2L^5\). Now identity \(L_{x(x^2x^2)} = 0\) and relations (2.5) and (2.14) imply \(L^2UL = -L^3U - 4L^5\). Thus, we have the following multiplication identities.

\[
\begin{array}{cccc}
\text{Table vi. Multiplication identities of degree 5.} \\
\hline
ULU & LUL^2 & L^3U & L^5 \\
\hline
ULU & 0 & 2 & 0 & 0 \\
LUU & 0 & -2 & -2 & -4 \\
L^2UL & 0 & 0 & -1 & -4 \\
UL^3 & 0 & 0 & 0 & 2 \\
\end{array}
\]

From Table ii, we can prove that

\[(UL)^2 = -UL^2U - (LU)^2 + 2L^3UL + 4L^4U + 16L^6,\]

and \(\delta[x^2] \{ x^2(x(xy)) - 2x(x(x(xy))) \} = 0 \) forces

\[(UL)^2 + ULL^2U + 2L^3UL + 4L^6 = 0.\]

Combining (2.15) and (2.16), we have \((LU)^2 = 4L^6\) and \((UL)^2 = -UL^2U + 2L^4U + 4L^6\). Now, we can check easily the following multiplication identities.

\[
\begin{array}{ccc}
\text{Table vii. Multiplication identities of degree 6.} \\
\hline
UUU & L^4U & L^6 \\
\hline
UUU & -2 & 4 & 8 \\
UULL & 0 & 0 & 4 \\
ULUL & -1 & 2 & 4 \\
LUUL & 0 & 2 & 0 \\
LULU & 0 & 0 & 4 \\
\end{array}
\quad
\begin{array}{ccc}
\hline
ULU & L^4U & L^6 \\
\hline
LLU & 0 & -4 & -4 \\
UL^4 & 0 & 0 & 2 \\
LUL^3 & 0 & 0 & 2 \\
L^2UL^2 & 0 & 1 & 0 \\
L^3UL & 0 & -1 & -4 \\
\end{array}
\]

**Theorem 2.** Let \(A\) be an algebra over a field \(F\) of characteristic \(\neq 2\) or 3, satisfying the identities \(x^4 = 0\) and \(x(x^2x^2) = 0\). Then every monomial in \(P\) of \(x\)-degree \(\geq 7\) and \(y\)-degree \(1\) is an identity in \(A\). In particular, \(L^7_a = 0\) for all \(a \in A\). Furthermore, the algebra generated by \(L_x\) and \(L_{x^2}\) is spanned, as vector space, by \(L, U, L^2, UL, LU, L^3, UL^2, UL, L^2U, L^4, ULU, LUL^2, L^3U, L^5, UL^2U, L^4U, L^6\).

**Proof.** We shall prove that every monomial of \(x\)-degree \(\geq 7\) and \(y\)-degree \(1\) is an identity in \(A\). Multiplying the operators in the first line of Table vii with \(L\) and \(U\) from the left and from the right, and the operators in the first line of Table vi with \(U\) from the left and from the right, and next using reductions from Tables i-vii we see that we only need to prove that \(LUL^2U = 0, L^5U = 0\) and \(L^7 = 0\) are multiplication identities in \(A\). Now, we have 0 = \(\delta[y, x^2x^2] \{ x(x^2) \} = 4L_{x^2x^2}UL + 4LU_{x^2x^2} = -16LULUL - 16LULUL = -32LULUL = -32(LU)^2L = -2L^7\), so that \(L^7 = 0\). Also 0 = \(LULUL = L(UL)^2 = -LUL^2U + 2L^5U\). Therefore, \(LUL^2U = 2L^5U\). Finally, from Table vi we have that 0 = \((L^2UL + L^3U + 4L^5)\) \(L^2 = L^2UL^3 + L^3UL^2 = L^3UL^2 = L(L^2UL^2) = L^5U\). This proves the theorem. \(\Box\)
2.2. The identity \( x(x((xx)(xx)))=0 \)

In this subsection we consider the class of all algebras in \( V \) satisfying the identity \( x(x(xx)) = 0 \). Because we use linearization process of identities and \( x(x^2x^2) \) has degree 6, we need consider the field \( F \) of characteristic not 5 (2 or 3.)

From linearization \( \delta[y]\{x(x(xx))\} \), we get the multiplication identity \( L_x(x^2x^2)+LL_x^2x^2+4L^2UL = 0 \) and now Lemma 2 forces

\[
LUU = -2LU^2 - 2L^3U - 4L^5.
\]

The relation \( 0 = \delta[y, x^2]\{x(x^2x^2)\} = UL_x^2x^2 + 4LL^2x^3L + 4ULUL + 4LUUL + 8L^2Lx_3L + 4L^2UU \) implies

\[
LUL^3 = -\frac{1}{2}(L^2UL^2 + L^3UL),
\]

since we can use identities from Tables i-v. Next, by \( 0 = \delta[y, x^2]\{x(x^2x^2)\} \) and \( 0 = \delta[y, x^2, x^2]\{x(x^2x^2)\} \) we get

\[
[\text{cl}]L^4UL = -3L^5U - 16L^7,
\]

\[
L^2ULU = -L^3UL^2 + 5L^5U + 28L^7,
\]

and identities \( 0 = \delta[y, x^2, x, x]\{x(x^2x^2)\} \) and \( 0 = \delta[y, x^2, x^3]\{x(x^2x^2)\} \) imply

\[
[\text{cl}]L^4U = \frac{1}{2}L^2UL^2U + 24L^6U + 62L^8,
\]

\[
L^2UL^2U = 48L^6U + 156L^8.
\]

Now, identity \( 0 = \delta[y, xx^2]\{xx^3\} \) forces

\[
L^6U = -2L^8.
\]

**Theorem 3.** Let \( A \) be a commutative algebra over a field \( F \) of characteristic not 2, 3 or 5, satisfying the identities \( x^4 = 0 \) and \( x(x^2x^2) = 0 \). Then every monomial in \( P \) of \( x \)-degree \( \geq 9 \) and \( y \)-degree 1 is an identity in \( A \). In particular, \( L_9^a = 0 \) for all \( a \in A \).

**Proof.** By Tables i-v, we only need to prove that \( LUL^4U = 0, L^2UL^2UL = 0, L^7U = 0 \) and \( L^9 = 0 \) are multiplication identities in \( A \). From (2.19-2.23) may be deduced immediately \( L^7U = -2L^9 \) and \( 2L^9 = 2L^8L = -L^6UL = -L^2(L^4UL) = 3L^7U + 16L^9 = -6L^9 + 16L^9 = 10L^9 \). Therefore \( L^9 = 0 \) and \( L^7U = 0 \) are identities in \( A \). Now \( L^2UL^2UL = (L^2UL^2U)L = 48L^6UL + 156L^9 = 0 \) and \( LUL^4U = L(U^4L) = -(1/2)L^3UL^2U + 24L^7U + 62L^9 = -(1/2)L(L^2UL^2U) = -24L^7U - 78L^9 = 0. \) This proves the theorem.  \( \square \)
References


Juan C. Gutierrez Fernandez
Departamento de Matemtica-IME,
Universidade de So Paulo,
Caixa Postal 66281,
CEP 05315-970,
So Paulo, SP,
Brazil
e-mail : jcgf@ime.usp.br