A NEW NOTION OF SP-COMPACT _L_-FUZZY SETS *

SHI - ZHONG BAI
WUYI UNIVERSITY, P. R. CHINA

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Abstract

A new notion of SP-compactness is introduced in _L_-topological spaces by means of semi-preopen _L_-sets and their inequality, where _L_ is a complete De Morgan algebra. This new notion does not depend on the structure of basis lattice _L_ and _L_ does not require any distributivity. This new notion implies semicompactness, hence it also implies compactness. This new notion is a good extension and it has many characterizations if _L_ is completely distributive De Morgan algebra.

Key Words and Phrases: _L_-topology; semi-preopen _L_-set; SP-compactness.

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Dedicator: Department of Mathematics, Wuyi University, Guangdong 529020, P.R.China

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1. Introduction

In general topological spaces, the concepts of semi-preopen sets and semi-preclosed sets were introduced by Andrijevic [1]. Thakur and Singh extended these concepts to \([0,1]\)-topological spaces [11] in the Chang’s [4] sense. In [2], we introduced the concept of SP-compactness in \(L\)-topological spaces. It preserves many good properties of compactness in general topological spaces. However, the SP-compactness relies on the structure of basis lattice \(L\) and \(L\) is required to be completely distributive. In [10], a new definition of fuzzy compactness is presented in \(L\)-topological spaces by means of open \(L\)-sets and their inequality, where \(L\) is a complete De Morgan algebra. This new definition doesn’t depend on the structure of \(L\).

In this paper, following the lines of [10], we’ll introduce a new notion of SP-compactness in \(L\)-topological spaces by means of semi-preopen \(L\)-sets and their inequality. It is a strong form of semi-compactness [8], hence it is also a strong form of compactness [10]. It can also be characterized by semi-preclosed \(L\)-sets and their inequality. It is defined for any \(L\)-subset, and it is hereditary for semi-preclosed subsets, finitely additive, and is preserved under SP-irresolute mapping. This new form of SP-compactness is a good extension and it has many characterizations when \(L\) is completely distributive De Morgan algebra.

2. Preliminaries

Throughout this paper, \((L, \lor, \land, ')\) is a complete De Morgan algebra, \(X\) a nonempty set. \(L^X\) is the set of all \(L\)-fuzzy sets (or \(L\)-sets for short) on \(X\). The smallest element and the largest element in \(L^X\) are denoted by \(\underline{0}\) and \(\underline{1}\). An element \(a\) in \(L\) is called prime element if \(b \land c \leq a\) implies that \(b \leq a\) or \(c \leq a\). \(a\) in \(L\) is called a co-prime element if \(a'\) is a prime element [6]. The set of nonunit prime elements in \(L\) is denoted by \(P(L)\). The set of nonzero co-prime elements in \(L\) is denoted by \(M(L)\).

The binary relation \(<\) in \(L\) is defined as follows: for \(a, b \in L, a < b\) iff for every subset \(D \subseteq L\), the relation \(b \leq \sup D\) always implies the existence of \(d \in D\) with \(a \leq d\) [5]. In a completely distributive De Morgan algebra \(L\), each element \(b\) is a sup of \(\{a \in L | a < b\}\). \(\{a \in L | a < b\}\) is called the greatest minimal family of \(b\) in the sense of [7,12], in symbol \(\beta(b)\). Moreover for \(b \in L\), define \(\beta^*(b) = \beta(b) \cap M(L)\), \(\alpha(b) = \{a \in L | a' < b\}\) and \(\alpha^*(b) = \alpha(b) \cap P(L)\). For \(a \in L\) and \(A \in L^X\), we denote \(A^{(a)} = \{x \in X | A(x) \nleq a\}\) and \(A_{(a)} = \{x \in X | a \in \beta(A(x))\}\). For a subfamily \(\psi \subseteq L^X\), \(2(\psi)\) denotes
the set of all finite subfamilies of \( \psi \).

An \( L \)-topological space (or \( L \)-ts for short) is a pair \((X, \delta)\), where \( \delta \) is a subfamily of \( L^X \) which contains 0,1 and is closed for any suprema and finite infima. \( \delta \) is called an \( L \)-topology on \( X \). Each member of \( \delta \) is called an open \( L \)-set and its quasi-complement is called a closed \( L \)-set. The semi-preopen set and semi-preclosed set are defined in \([0,1]\)-topological space in [11]. Analogously we can generalize it to \( L \)-subset in \( L \)-topological spaces. Let \((L^X, \delta)\) be an \( L \)-ts. \( A \in L^X \) is called semi-preopen if there is a preopen set \( B \) such that \( B \leq A \leq B^- \), and semi-preclosed if there is a preclosed set \( B \) such that \( B^o \leq A \leq B \), where \( B^o \) and \( B^- \) are the interior and closure of \( B \), respectively.

**Definition 2.1** ([7,12]). For a topological space \((X, \tau)\), let \( \omega_L(\tau) \) denote the family of all the lower semi-continuous maps from \((X, \tau)\) to \( L \), that is, \( \omega_L(\tau) = \{ A \in L^X | A^{(a)} \in \tau, a \in L \} \). Then \( \omega_L(\tau) \) is an \( L \)-topology on \( X \), in this case, \((X, \omega_L(\tau))\) is topologically generated by \((X, \tau)\).

**Definition 2.2** ([7,12]). An \( L \)-ts \((X, \delta)\) is weak induced if for all \( a \in L \), for all \( A \in \delta \), it follows that \( A^{(a)} \in [\delta] \), where \([\delta]\) denotes the topology formed by all crisp sets in \( \delta \). It is obvious that \((X, \omega_L(\tau))\) is weak induced.

**Definition 2.3**([8,9]). Let \((X, \delta)\) be an \( L \)-ts, \( a \in L \setminus \{1\} \), and \( A \in L^X \). A family \( \mu \subseteq L^X \) is called

1. an \( a \)-shading of \( A \) if for any \( x \in X \), \( A'(x) \cup \bigvee_{B \in \mu} B(x) \not\leq a \).
2. a strong \( a \)-shading (briefly \( S \)-a-shading) of \( A \) if
   \[ \bigwedge_{x \in X} (A'(x) \cup \bigvee_{B \in \mu} B(x)) \not\leq a. \]
3. an \( a \)-R-neighborhood family (briefly \( a \)-R-NF) of \( A \) if for any \( x \in X \),
   \[ (A(x) \land \bigwedge_{B \in \mu} B(x)) \not\geq a. \]
4. a strong \( a \)-R-neighborhood family (briefly \( S \)-a-R-NF) of \( A \) if \( \bigvee_{x \in X} (A(x) \land \bigwedge_{B \in \mu} B(x)) \not\geq a. \)

It is obvious that an \( S \)-a-shading of \( A \) is an \( a \)-shading of \( A \), an \( S \)-a-R-NF of \( A \) is an \( a \)-R-NF of \( A \), and \( \mu \) is an \( S \)-a-R-NF of \( A \) iff \( \mu' \) is an \( S \)-a-shading of \( A \).

**Definition 2.4**([8]). Let \((X, \delta)\) be an \( L \)-ts, \( a \in L \setminus \{0\} \) and \( A \in L^X \). A family \( \mu \subseteq L^X \) is called

1. a \( \beta_a \)-cover of \( A \) if for any \( x \in X \), it follows that \( a \in \beta(A'(x) \cup \bigvee_{B \in \mu} B(x)) \).
(2) a strong $\beta_a$-cover (briefly S-$\beta_a$-cover) of $A$ if $a \in \beta(\bigwedge_{x \in X}(A'(x) \vee \bigvee_{B \in \mu} B(x)))$.

(3) a $Q_a$-cover of $A$ if for any $x \in X$, it follows that $A'(x) \vee \bigvee_{B \in \mu} B(x) \geq a$.

It is obvious that an S-$\beta_a$-cover of $A$ must be a $\beta_a$-cover of $A$, and a $\beta_a$-cover of $A$ must be a $Q_a$-cover of $A$.

**Definition 2.5** ([8,9]). Let $a \in L \setminus \{0\}$ and $A \in L^X$. A family $\mu \subseteq L^X$ is said to have weak $a$-nonempty intersection in $A$ if $\bigvee_{x \in X}(A(x) \wedge \bigwedge_{B \in \mu} B(x)) \geq a$. $\mu$ is said to have the finite weak $a$-intersection property in $A$ if every finite subfamily $\nu$ of $\mu$ has weak $a$-nonempty intersection in $A$.

**Lemma 2.6** ([8]). Let $L$ be a complete Heyting algebra, $f : X \to Y$ be a map and $f_L^{-} : L^X \to L^Y$ is the extension of $f$, then for any family $\psi \subseteq L^Y$,

$$\bigvee_{y \in Y}(f_L^{-}(A)(y) \wedge \bigwedge_{B \in \psi} B(y)) = \bigvee_{x \in X}(A(x) \wedge \bigwedge_{B \in \psi} f_L^{-}(B)(x)).$$

**Definition 2.7** ([2,3,11]). Let $(X,\delta)$ and $(Y,\tau)$ be two $L$-ts’s. A map $f : (X,\delta) \to (Y,\tau)$ is called

(1) semi-precontinuous if $f_L^{-}(B)$ is semi-preopen in $(X,\delta)$ for every $B \in \tau$.

(2) semi-preirresolute if $f_L^{-}(B)$ is semi-preopen in $(X,\delta)$ for every semi-preopen $L$-set $B$ in $(Y,\tau)$.

3. **Definitions and properties**

**Definition 3.1.** Let $(X,\delta)$ be an $L$-ts. $A \in L^X$ is called SP-compact if for every family $\mu$ of semi-preopen $L$-sets, it follows that

$$\bigwedge_{x \in X}(A'(x) \vee \bigvee_{B \in \mu} B(x)) \leq \bigvee_{\mu \subseteq 2^{(\mu)}} \bigwedge_{x \in X}(A'(x) \vee \bigvee_{B \in \mu} B(x)).$$

$(X,\delta)$ is called SP-compact if 1 is SP-compact.

**Example 3.2.** Let $X = \{x\}$ and $L = \{0, 1/3, 2/3, 1\}$. For each $a \in L$ define $a' = 1 - a$. Let $\delta = \{\emptyset, A, X\}$, where $A(x) = 2/3$, then $\delta$ is a topology on $X$. Clearly, any $L$-set in $(X,\delta)$ is SP-compact.

**Example 3.3.** Let $X$ be an infinite set (or $X$ be a singleton), $A$ and $C$ be two $[0, 1]$-sets on $X$ defined as $A(x) = 0.5$, for all $x \in X$; $C(x) = 0.6$, for all $x \in X$. Take $\delta = \{\emptyset, A, X\}$, then $\delta$ is a topology on $X$. Obviously, any
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$[0,1]$-set in $(X, \delta)$ is semi-preopen, and the set of all semi-open $[0,1]$-sets in $(X, \delta)$ is $\delta$. In this case, we easily obtain that $C$ is not SP-compact, and any $[0,1]$-set in $(X, \delta)$ is semi-compact.

**Remark 3.4.** Since every semi-open $L$-set is semi-preopen[2,11], every SP-compact $L$-set is semi-compact. Example 3.3 shows that semi-compact $L$-set needn’t be SP-compact.

**Theorem 3.5.** Let $(X, \delta)$ be an $L$-ts. $A \in L^X$ is SP-compact iff for every family $\mu$ of semi-preclosed $L$-sets, it follows that

$$\bigvee_{x \in X} (A(x) \land \bigwedge_{B \in \mu} B(x)) \geq \bigwedge_{\nu \in 2^{\mu}} \bigvee_{x \in X} (A(x) \land \bigwedge_{B \in \nu} B(x)).$$

**Proof.** This is immediate from Definition 3.1 and quasi-complement.

**Theorem 3.6.** Let $(X, \delta)$ be an $L$-ts and $A \in L^X$. Then the following conditions are equivalent.

1. $A$ is SP-compact.
2. For any $a \in L \setminus \{1\}$, each semi-preopen S-a-shading $\mu$ of $A$ has a finite subfamily which is an S-a-shading of $A$.
3. For any $a \in L \setminus \{0\}$, each semi-preclosed S-a-R-NF $\psi$ of $A$ has a finite subfamily which is an S-a-R-NF of $A$.
4. For any $a \in L \setminus \{0\}$, each family of semi-preclosed $L$-sets which has the finite weak $a$-intersection property in $A$ has weak $a$-nonempty intersection in $A$.

**Proof.** This is immediate from Definition 3.1 and Theorem 3.5.

**Theorem 3.7.** Let $L$ be a complete Heyting algebra. If both $C$ and $D$ are SP-compact, then $C \vee D$ is SP-compact.

**Proof.** For any family $\mu$ of semi-preclosed $L$-sets, by Theorem 3.5 we have that

$$\bigvee_{x \in X} ((C \vee D)(x) \land \bigwedge_{B \in \mu} B(x))$$

$$= \left\{ \bigvee_{x \in X} (C(x) \land \bigwedge_{B \in \mu} B(x)) \right\} \lor \left\{ \bigvee_{x \in X} (D(x) \land \bigwedge_{B \in \mu} B(x)) \right\}$$

$$\geq \left\{ \bigwedge_{\nu \in 2^{\mu}} \bigvee_{x \in X} (C(x) \land \bigwedge_{B \in \nu} B(x)) \right\} \lor \left\{ \bigwedge_{\nu \in 2^{\mu}} \bigvee_{x \in X} (D(x) \land \bigwedge_{B \in \nu} B(x)) \right\}$$

$$= \bigwedge_{\nu \in 2^{\mu}} \bigvee_{x \in X} ((C \vee D)(x) \land \bigwedge_{B \in \nu} B(x)).$$

This shows that $C \vee D$ is SP-compact.
**Theorem 3.8.** If $C$ is SP-compact and $D$ is semi-preclosed, then $C \land D$ is SP-compact.

**Proof.** For any family $\mu$ of semi-preclosed $L$-sets, by Theorem 3.5 we have that
\[
\bigvee_{x \in X} \left( (C \land D)(x) \land \bigwedge_{B \in \mu} B(x) \right)
= \bigwedge_{\nu \in \mathbb{2}^{\mathbb{2}^{\nu}(D)}} \bigvee_{x \in X} \left( C(x) \land \bigwedge_{B \in B(x)} B(x) \right)
\geq \bigwedge_{\nu \in \mathbb{2}^{\mathbb{2}^{\nu}(D)}} \bigvee_{x \in X} \left( C(x) \land \bigwedge_{B \in B(x)} B(x) \right)
= \bigwedge_{\nu \in \mathbb{2}^{\mathbb{2}^{\nu}(D)}} \bigvee_{x \in X} \left( C(x) \land D(x) \land \bigwedge_{B \in B(x)} B(x) \right)
\]

This shows that $C \land D$ is SP-compact.

**Corollary 3.9.** Let $(X, \delta)$ be SP-compact and $D \in L^X$ is semi-preclosed. Then $D$ is SP-compact.

**Definition 3.10.** Let $(X, \delta)$ and $(Y, \tau)$ be two $L$-ts’s. A map $f : (X, \delta) \to (Y, \tau)$ is called

(1) strongly semi-precontinuous if $f^{-1}_{\mathbb{L}}(B)$ is semi-preopen in $(X, \delta)$ for every semi-open $L$-set $B$ in $(Y, \tau)$.

(2) strongly semi-preirresolute if $f^{-1}_{\mathbb{L}}(B)$ is semi-open in $(X, \delta)$ for every semi-preopen $L$-set $B$ in $(Y, \tau)$.

**Remark 3.11.** It is obvious that a strongly semi-precontinuous map is semi-precontinuous, and a strongly semi-preirresolute map is semi-preirresolute. None of the converses need be true.

**Example 3.12.** Let $X = \{x, y\}, L = [0, 1], \forall a \in L, a' = 1 - a$, and $A, B, C, D \in L^X$ defined as follows:

- $A(x) = 0.2, A(y) = 0.1$;
- $B(x) = 0.5, B(y) = 0.5$;
- $C(x) = 0.3, C(y) = 0.2$;
- $D(x) = 0.6, D(y) = 0.7$.

Then $\delta = \{0, A, B, 1\}$ and $\tau = \{0, C, 1\}$ are topologies on $X$. Let $f : (X, \delta) \to (X, \tau)$ be an identity mapping. Obviously, $f$ is semi-precontinuous.
We can easily get that $D$ is a semiopen $L$-set in $(X, \tau)$ and that $f_L^{-1}(D)$ is not semi-preopen in $(X, \delta)$. Thus, $f$ is not strongly semi-precontinuous.

**Example 3.13.** Let $X = \{x, y\}$, $L = [0, 1]$, $\forall a \in L, a' = 1 - a$, and $A, B, C \subseteq L^X$ defined as follows:

$A(x) = 0.5, \quad A(y) = 0.5; \\
B(x) = 0.7, \quad B(y) = 0.6; \\
C(x) = 0.7, \quad C(y) = 0.8.$

Then $\delta = \{0, A, 1\}$ and $\tau = \{0, B, 1\}$ are topologies on $X$. Let $f : (X, \delta) \to (X, \tau)$ be an identity mapping. Obviously, $f$ is semi-preirresolute. We can easily get that $C$ is a semi-preopen $L$-set in $(X, \tau)$ and that $f_L^{-1}(C)$ is not semiopen in $(X, \delta)$. Thus, $f$ is not strongly semi-preirresolute.

**Theorem 3.14.** Let $L$ be a complete Heyting algebra and $f : (X, \delta) \to (Y, \tau)$ be a semi-preirresolute map. If $A$ is an SP-compact $L$-set in $(X, \delta)$, then so is $f_L^{-1}(A)$ in $(Y, \tau)$.

**Proof.** Suppose that $\mu$ is a family of semi-preclosed $L$-sets in $(Y, \tau)$, by Lemma 2.6 and SPR-compactness of $A$, we have that

$$\bigvee_{y \in Y} (f_L^{-1}(A)(y) \wedge \bigwedge_{B \in \mu} B(y))$$

$$= \bigvee_{x \in X} (A(x) \wedge \bigwedge_{B \in \mu} f_L B(x))$$

$$\geq \bigwedge_{\nu \in 2^{(\mu)}} \bigvee_{x \in X} (A(x) \wedge \bigwedge_{B \in \nu} f_L^{-1} B(x))$$

$$= \bigwedge_{\nu \in 2^{(\mu)}} \bigvee_{y \in Y} (f_L^{-1}(A)(y) \wedge \bigwedge_{B \in \nu} B(y)).$$

Therefore $f_L^{-1}(A)$ is SP-compact.

Analogously, we can obtain the following theorems.

**Theorem 3.15.** Let $L$ be a complete Heyting algebra and $f : (X, \delta) \to (Y, \tau)$ be a semi-precontinuous map. If $A$ is an SP-compact $L$-set in $(X, \delta)$, then $f_L^{-1}(A)$ is a compact $L$-set in $(Y, \tau)$.

**Theorem 3.16.** Let $L$ be a complete Heyting algebra and $f : (X, \delta) \to (Y, \tau)$ be a strongly semi-precontinuous map. If $A$ is an SP-compact $L$-set in $(X, \delta)$, then $f_L^{-1}(A)$ is a semi-compact $L$-set in $(Y, \tau)$.

**Theorem 3.17.** Let $L$ be a complete Heyting algebra and $f : (X, \delta) \to (Y, \tau)$ be a strongly semi-preirresolute map. If $A$ is a semi-compact $L$-set
in \((X, \delta)\), then \(f^-_L(A)\) is an SP-compact \(L\)-set in \((Y, \tau)\).

4. Further properties and goodness

In this section, we assume that \(L\) is a completely distributive de Morgan algebra.

**Theorem 4.1.** Let \((X, \delta)\) be an \(L\)-ts and \(A \in L^X\). Then the following conditions are equivalent.

1. \(A\) is SP-compact.
2. For any \(a \in L \setminus \{0\}\), each semi-preclosed \(S\)-\(a\)-R-NF \(\psi\) of \(A\) has a finite subfamily which is an \(S\)-\(a\)-R-NF of \(A\).
3. For any \(a \in L \setminus \{0\}\), each semi-preclosed \(S\)-\(a\)-R-NF \(\psi\) of \(A\) has a finite subfamily which is an \(a\)-R-NF of \(A\).
4. For any \(a \in L \setminus \{0\}\) and any semi-preclosed \(S\)-\(a\)-R-NF \(\psi\) of \(A\), there exist a finite subfamily \(\varphi\) of \(\psi\) and \(b \in \beta(a)\) such that \(\varphi\) is an \(S\)-\(b\)-R-NF of \(A\).
5. For any \(a \in L \setminus \{0\}\) and any semi-preclosed \(S\)-\(a\)-R-NF \(\psi\) of \(A\), there exist a finite subfamily \(\varphi\) of \(\psi\) and \(b \in \beta(a)\) such that \(\varphi\) is a \(b\)-R-NF of \(A\).
6. For any \(a \in L \setminus \{1\}\), each semi-preopen \(S\)-\(a\)-shading \(\mu\) of \(A\) has a finite subfamily which is an \(S\)-\(a\)-shading of \(A\).
7. For any \(a \in L \setminus \{1\}\), each semi-preopen \(S\)-\(a\)-shading \(\mu\) of \(A\) has a finite subfamily which is an \(a\)-shading of \(A\).
8. For any \(a \in L \setminus \{1\}\) and any semi-preopen \(S\)-\(a\)-shading \(\mu\) of \(A\), there exist a finite subfamily \(\nu\) of \(\mu\) and \(b \in \alpha(a)\) such that \(\nu\) is an \(S\)-\(b\)-shading of \(A\).
9. For any \(a \in L \setminus \{1\}\) and any semi-preopen \(S\)-\(a\)-shading \(\mu\) of \(A\), there exist a finite subfamily \(\nu\) of \(\mu\) and \(b \in \alpha(a)\) such that \(\nu\) is a \(b\)-shading of \(A\).
10. For any \(a \in L \setminus \{0\}\), each semi-preopen \(S\)-\(\beta_a\)-cover \(\mu\) of \(A\) has a finite subfamily which is an \(S\)-\(\beta_a\)-cover of \(A\).
11. For any \(a \in L \setminus \{0\}\), each semi-preopen \(S\)-\(\beta_a\)-cover \(\mu\) of \(A\) has a finite subfamily which is a \(\beta_a\)-cover of \(A\).
12. For any \(a \in L \setminus \{0\}\) and any semi-preopen \(S\)-\(\beta_a\)-cover \(\mu\) of \(A\), there exist a finite subfamily \(\nu\) of \(\mu\) and \(b \in L\) with \(a \in \beta(b)\) such that \(\nu\) is an \(S\)-\(\beta_b\)-cover of \(A\).
13. For any \(a \in L \setminus \{0\}\) and any semi-preopen \(S\)-\(\beta_a\)-cover \(\mu\) of \(A\), there exist a finite subfamily \(\nu\) of \(\mu\) and \(b \in L\) with \(a \in \beta(b)\) such that \(\nu\) is a \(\beta_b\)-cover of \(A\).
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(14) For any \( a \in L \setminus \{0\} \) and any \( b \in \beta(a) \setminus \{0\} \), each semi-preopen \( Q_a \)-cover \( \mu \) of \( A \) has a finite subfamily which is a \( Q_b \)-cover of \( A \).

(15) For any \( a \in L \setminus \{0\} \) and any \( b \in \beta(a) \setminus \{0\} \), each semi-preopen \( Q_a \)-cover \( \mu \) of \( A \) has a finite subfamily which is a \( \beta_b \)-cover of \( A \).

(16) For any \( a \in L \setminus \{0\} \) and any \( b \in \beta(a) \setminus \{0\} \), each semi-preopen \( Q_a \)-cover \( \mu \) of \( A \) has a finite subfamily which is an \( S-\beta_b \)-cover of \( A \).

Proof. This is analogous to the proof of Theorem 5.3 in [8].

Remark 4.2. In Theorem 4.1, \( a \in L \setminus \{0\} \) and \( b \in \beta(a) \) can be replaced by \( a \in M(L) \) and \( b \in \beta^*(a) \) respectively. \( a \in L \setminus \{1\} \) and \( b \in \alpha(a) \) can be replaced by \( a \in P(L) \) and \( b \in \alpha^*(a) \) respectively. Thus, we can obtain other 15 equivalent conditions about the SP-compactness.

Lemma 4.3 Let \((X,\omega_L(\tau))\) be generated topologically by \((X,\tau)\). If \( A \) is a preopen \( L \)-set in \((X,\tau)\), then \( \chi_A \) is a preopen set in \((X,\omega_L(\tau))\). If \( B \) is a preopen \( L \)-set in \((X,\omega_L(\tau))\), then \( B(a) \) is a preopen set in \((X,\tau)\) for every \( a \in L \).

Proof. This is analogous to the proof of Theorem 5.7 in [9].

Lemma 4.4 Let \((X,\omega_L(\tau))\) be generated topologically by \((X,\tau)\). If \( A \) is a semi-preopen \( L \)-set in \((X,\tau)\), then \( \chi_A \) is a semi-preopen set in \((X,\omega_L(\tau))\). If \( B \) is a semi-preopen \( L \)-set in \((X,\omega_L(\tau))\), then \( B(a) \) is a semi-preopen set in \((X,\tau)\) for every \( a \in L \).

Proof. This is analogous to the proof Theorem 5.4 in [8], by Lemma 4.3.

Theorem 4.5. Let \((X,\tau)\) be a topological space and \((X,\omega_L(\tau))\) be generated topologically by \((X,\tau)\). Then \((X,\omega_L(\tau))\) is SP-compact iff \((X,\tau)\) is SP-compact.

Proof. Necessity. Let \( \mu \) be a semi-preopen cover of \((X,\tau)\). Then \( \{\chi_A | A \in \mu\} \) is a family of semi-preopen \( L \)-sets in \((X,\omega_L(\tau))\) with \( \bigwedge_{x \in X} (\bigvee_{A \in \mu} \chi_A(x)) = 1 \). From SP-compactness of \((X,\omega_L(\tau))\), we have that
\[
1 = \bigwedge_{x \in X} (\bigvee_{A \in \mu} \chi_A(x)) \leq \bigvee_{\nu \in 2(\mu)} \bigwedge_{x \in X} (\bigvee_{A \in \nu} \chi_A(x)).
\]
This implies that there exists \( \nu \in 2(\mu) \) such that \( \bigwedge_{x \in X} (\bigvee_{A \in \nu} \chi_A(x)) = 1 \). Hence, \( \nu \) is a cover of \((X,\tau)\). Therefore \((X,\tau)\) is SP-compact.
Sufficiency. Let $\mu$ be a family of semi-preopen $L$-sets in $(X, \omega_L(\tau))$ and $\bigwedge_{x \in X} (\bigvee_{B \in \mu} B(x)) = a$. If $a = 0$, obviously we have that $\bigwedge_{x \in X} (\bigvee_{B \in \mu} B(x)) \leq \bigvee_{\nu \in 2(\mu)} \bigwedge_{x \in X} B(x)$.

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a) \setminus \{0\}$, we have that $b \in \beta(\bigwedge_{x \in X} (\bigvee_{B \in \mu} B(x))) \subseteq \bigcap_{x \in X} \beta(\bigvee_{B \in \mu} B(x)) = \bigcup_{x \in X} \beta(B(x))$.

From Lemma 4.4, this implies that $\{B(b)|B \in \mu\}$ is a semi-preopen cover of $(X, \tau)$. From SP-compactness of $(X, \tau)$, we know that there exists $\nu \in 2(\mu)$ such that $\{B(b)|B \in \nu\}$ is a cover of $(X, \tau)$. Hence $b \leq \bigwedge_{x \in X} (\bigvee_{B \in \nu} B(x))$.

Further, we have that $b \leq \bigwedge_{x \in X} (\bigvee_{B \in \nu} B(x)) \leq \bigvee_{\nu \in 2(\mu)} \bigwedge_{x \in X} \bigvee_{B \in \nu} B(x)$.

This implies that $\bigwedge_{x \in X} (\bigvee_{B \in \mu} B(x)) = a = \bigvee\{b|b \in \beta(a)\} \leq \bigvee_{\nu \in 2(\mu)} \bigwedge_{x \in X} \bigvee_{B \in \nu} B(x)$.

Therefore, $(X, \omega_L(\tau))$ is SP-compact.

References

A new notion of SP-compact L-fuzzy sets


Shi-Zhong Bai
Department of Mathematics
Wuyi University
Guangdong 529020
P.R. China
E-mail: shizhongbai@yahoo.com