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GALERKIN APPROXIMATION FOR A SEMI LINEAR PARABOLIC PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract

We analyze a θ -method for the numerical solution of a semi linear parabolic problem with boundary conditions containing integrals over the interior of the interval. Existence and convergence are proved for $\theta \geq 1/2$, numerical application is given.

Keywords : *Semi linear parabolic problem, nonlocal boundary conditions, θ -Galerkin method.*

AMS classification : *65M60, 65M15, 35K55.*

1. Introduction

This paper is concerned with the numerical solution of the semi linear parabolic problem

$$(1.1) \quad \begin{cases} u_t - (a(x)u_x)_x - b(x)u_x = F(u, x, t), \\ \alpha < x < \beta, \quad 0 < t \leq T, \end{cases}$$

subject to the nonlocal boundary conditions

$$(1.2) \quad \begin{cases} u(\alpha, t) = \int_{\alpha}^{\beta} K_0(x)u(x, t)dx + g_0(t), \\ u(\beta, t) = \int_{\alpha}^{\beta} K_1(x)u(x, t)dx + g_1(t), \\ 0 \leq t \leq T, \end{cases}$$

and the initial condition

$$(1.3) \quad u(x, 0) = u_0(x), \quad \alpha \leq x \leq \beta.$$

We assume that the function F is Lipschitz continuous on compact sets of $\mathbf{R}$, the functions a and b_x are supposed to be continuous and that there exist positive constants a_1, a_2, b_1, b_2 such that

$$(1.4) \quad a_1 \leq a(x) \leq a_2 \quad \text{and} \quad b_1 \leq b_x(x) \leq b_2, \quad \alpha \leq x \leq \beta.$$

This kind of initial-boundary-value problem arises in the quasi-static theory of thermoelasticity [2, 3]. The existence, uniqueness and some analytic properties of the solution of (1.1)-(1.3), have been studied by Day [2, 3], Friedman [6], Kawohl [7], when $b(x) = 0, \alpha = 0$ and $\beta = 1$ under the hypothesis that

$$(1.5) \quad \int_0^1 |K_0(x)|dx < 1, \quad \int_0^1 |K_1(x)|dx < 1.$$

We assume that the functions u, u_0, g_0, g_1 are sufficiently smooth and the functions K_0 and K_1 satisfy

$$(1.6) \quad \begin{aligned} (\beta - \alpha) \int_{\alpha}^{\beta} |K_0(x)|^2 dx &\leq \mu^2 < 1, \\ (\beta - \alpha) \int_{\alpha}^{\beta} |K_1(x)|^2 dx &\leq \mu^2 < 1. \end{aligned}$$

Fairweather and López-Marcos [5] considered the Crank-Nicholson method and the extrapolated Crank-Nicholson method for the problem (1.1)-(1.3)

when $b(x) = 0$, $\alpha = 0$, $\beta = 1$ using an energy argument, they proved the convergence of both methods under the condition (1.6), Fairweather, López-Marcos and A.Boutayeb [11] analyzed an orthogonal spline collocation method for a quasi linear parabolic problem, recently the present authors [1] considered several methods of extrapolations for a linear parabolic problem with non local boundary conditions.

The main purpose of this paper is to show the convergence of a θ -finite element Galerkin method (θ -method) for the solution of (1.1-1.3), and for $\theta \in [\frac{1}{2}, 1]$, using similar energy argument to the one in [5]. This paper is outlined as follows: Section 2 is devoted to the θ - method while the consistency, the stability, the convergence and the existence of the θ -method are treated in section 3, 4 and 5. In the last section we give some numerical results.

2. The θ -method

According to [5], let be $(\pi_h) = (x_j)_{j=0}^J$ a partition of $[\alpha, \beta]$ such that $\alpha = x_0 < x_1 < \dots < x_J = \beta$, where $h = \max_j(h_j)$, $h_j = x_j - x_{j-1}$ and $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq J$, we consider a family of partitions $(\pi_h)_{h \in H}$ where H is a set of positif numbers with $\inf(H) = 0$, we define the space S_h in the following way

$$S_h = \{v/v \in C^n[\alpha, \beta] \text{ and } v/I_j \in \mathbf{P}_r(I_j), 1 \leq j \leq J\},$$

and S_h° , the subspace of S_h , given by

$$S_h^\circ = \{v \in S_h / V(\alpha) = V(\beta) = 0\}.$$

Where $\mathbf{P}_r(E)$ denotes the set of polynomials of degree at most r , $r \geq 2$ for a closed interval E and $n < r$.

We denote by k the time step, $t_m = mk$, $m = 0, \dots, M = [T/k]$, $t_{m+\theta} = (m + \theta)k$, and $(\cdot, \cdot)$ the usual inner product on $L^2([\alpha, \beta])$.

The θ -Galerkin approximation (θ - method), $(U^m)_{m=0}^M \subset S_h$ to the solution of (1.1-1.3) is defined by

$$\begin{aligned} \left(\frac{U^{m+1} - U^m}{k}, v \right) + \left(a U_x^{m+\theta}, v_x \right) - \left(b U_x^{m+\theta}, v \right) - \left(F(U^{m+\theta}, t_{m+\theta}), v \right) &= 0 \\ v \in S_h^0, & \\ (2.1) \end{aligned}$$

$0 \leq m \leq M - 1$, where $U^{m+\theta} = \theta U^{m+1} + (1 - \theta) U^m$.

With the nonlocal boundary conditions

$$(2.2) \quad \begin{aligned} U^{m+1}(\alpha) &= \int_{\alpha}^{\beta} K_0(x) U^{m+1}(x) dx + g_0(t_{m+1}), \\ U^{m+1}(\beta) &= \int_{\alpha}^{\beta} K_1(x) U^{m+1}(x) dx + g_1(t_{m+1}), \end{aligned}$$

$0 \leq m \leq M-1$, $U^0 \in S_h$ will be specified later.

For the analysis of this method we require to examine the consistency (sect.3) and the stability (sect.4) of the method, in order to prove the existence and convergence, according to the framework developed in [8, 10]. In the sequel C denotes a positive constant whose value is not the same on each occurrence and which is independent of h and k , we define the norms $\|\cdot\|$ and $\|\cdot\|_{\infty}$, by

$$\|v\| = \sqrt{(v, v)}, \quad \|v\|_{\infty} = \max_{\alpha \leq x \leq \beta} |v(x)|.$$

Let $p_h : H_0^1([\alpha, \beta]) \longrightarrow S_h^0$ be the operator defined by

$$(2.3) \quad (a(z - p_h(z))_x, v_x) - (b(z - p_h(z))_x, v) = 0, \quad v \in S_h^0,$$

we note that

$$(2.4) \quad \|z - p_h(z)\|_{\infty} \leq C \|z\|_{W^{r,\infty}(0,1)}, \quad z \in H_0^1([\alpha, \beta]),$$

where $H_0^1([\alpha, \beta])$ is the usual Sobolev space on

$$[\alpha, \beta], \quad \|v\|_{W^{r,\infty}(0,1)} = \sum_{i=0}^r \|v^{(i)}\|_{\infty} \quad (\text{ see for instance Wheeler [9]).}$$

We define on S_h^0 the linear form

$$(2.5) \quad \begin{aligned} \Gamma^{m+\theta}(v, w)(\chi) &= \left(\frac{v-w}{k}, \chi\right) + (a(\theta v_x + (1-\theta)w_x), \chi_x) \\ &- (b(\theta v_x + (1-\theta)w_x), \chi) - (F(\theta v + (1-\theta)w, t_{m+\theta}), \chi), \end{aligned}$$

and the operators L_0 and L_1

$$L_0(V) = V(\alpha) - \int_{\alpha}^{\beta} K_0(x) V(x) dx, \quad L_1(V) = V(\beta) - \int_{\alpha}^{\beta} K_1(x) V(x) dx.$$

We denote

$$X_{hk} = (S_h)^{M+1} \quad \text{and} \quad Y_{hk} = S_h \times \left((S_h^0)^*\right)^M \times \mathbf{R}^M \times \mathbf{R}^M$$

where $(S_h^0)^*$ is the dual space of S_h^0 . It is clear that

$$\dim X_{hk} = \dim Y_{hk}.$$

We also define the operator

$$\Phi_{hk} : X_{hk} \longrightarrow Y_{hk},$$

by the relations

$$\begin{aligned} \Phi_{hk}(V^0, \dots, V^M) &= \left(U^0 - V^0, \left(\Gamma^{m-1+\theta}(V^m, V^{m-1}) \right)_{m=1}^M \right. \\ &\quad \left(L_0 \left(\frac{V^m - V^{m-1}}{k} \right) - \frac{g_0(t_m) - g_0(t_{m-1})}{k} \right)_{m=1}^M, \\ (2.6) \quad &\quad \left. \left(L_1 \left(\frac{V^m - V^{m-1}}{k} \right) - \frac{g_1(t_m) - g_1(t_{m-1})}{k} \right)_{m=1}^M \right), \end{aligned}$$

given an initial condition $U^0 \in S_h$ that satisfies

$$(2.7) \quad L_0(U^0) = g_0(\alpha), \quad L_1(U^0) = g_1(\alpha).$$

It is easy to show that an element $(U^0, \dots, U^M) \in (S_h)^{M+1}$ is a solution of the θ -method if and only if

$$(2.8) \quad \Phi_{hk}(U^0, \dots, U^M) = 0,$$

we also need in the analysis the norm

$$\|\Gamma\|_* = \sup_{\chi \in S_h^0} \frac{|\Gamma(\chi)|}{\|\chi\|} \quad \chi \neq 0.$$

It is well known that if K_0 and K_1 are zero then the θ -method is convergent for $\theta \geq \frac{1}{2}$ and the order of convergence is $O(h^r + k^l)$, where $l = 1$ if $\theta > \frac{1}{2}$, and $l = 2$ if $\theta = \frac{1}{2}$, our aim is to show that a similar result holds in the case of nonlocal boundary conditions.

3. Consistency

We consider the problem

$$\begin{cases} (a(x)(v_i)_x)_x - b(x)(v_i)_x = 0, & i = 0, 1, \\ v_0(\alpha) = 1, v_0(\beta) = 0, v_1(\alpha) = 0, v_1(\beta) = 1, \end{cases}$$

Whose solutions are the positive functions

$$v_0(x) = 1 - v_1(x), \quad v_1(x) = \frac{\phi(x)}{\phi(\beta)}$$

where

$$\phi(x) = \int_{\alpha}^x \frac{1}{a(u)} \left[\exp\left(\int_{\alpha}^u \frac{b(t)}{a(t)} dt\right) \right] du,$$

and the functions $\phi_0, \phi_1 \in S_h$, defined by

$$(3.1) \quad \begin{cases} (a(x)(\phi_i)_x, \chi_x) - (b(x)(\phi_i)_x, \chi) = 0, & i = 0, 1, \quad \chi \in S_h^0, \\ \phi_0(\alpha) = 1, \phi_0(\beta) = 0, \phi_1(\alpha) = 0, \phi_1(\beta) = 1, \end{cases}$$

Clearly ϕ_0 and ϕ_1 are the elliptic projections of the functions v_0 and v_1 , respectively, and from (2.4) we have

$$(3.2) \quad \|\phi_i - v_i\|_{\infty} \leq Ch^r,$$

it is easy to show that

$$(3.3) \quad (v_0, v_1) = \frac{1 - \|v_0\|^2 - \|v_1\|^2}{2},$$

$$(3.4) \quad \|Av_0 + Bv_1\|_{\infty} = \max(|A|, |B|),$$

for arbitrary constants $A, B \in \mathbf{R}$.

Furthermore, for $\nu > 0$ there exists $h_0 > 0$ such that for each $h \leq h_0$,

$$(3.5) \quad (|\phi_0|, |\phi_1|) \leq \frac{1 - \|\phi_0\|^2 - \|\phi_1\|^2 + \nu}{2},$$

and

$$(3.6) \quad \|A\phi_0 + B\phi_1\|_{\infty} \leq (1 + \nu)\max(|A|, |B|).$$

For $u(x, t)$ the solution of the problem (1.1-1.3) we define $\hat{u}(x, t) \in H_1^0([\alpha, \beta])$ by

$$(3.7) \quad \hat{u}(x, t) = u(x, t) - u(\alpha, t)v_0(x) - u(\beta, t)v_1(x),$$

and $\tilde{u}_h(x, t) \in S_h$ by

$$(3.8) \quad \tilde{u}_h(x, t) = P_h(\hat{u}(x, t)) - u(\alpha, t)\phi_0(x) - u(\beta, t)\phi_1(x),$$

we note that

$$(3.9) \quad p_h(\hat{u}_t(t)) = (p_h(\hat{u}(t)))_t.$$

We set

$$(3.10) \quad \tilde{u}_h^m = \tilde{u}_h(t_m), \quad 0 \leq m \leq M.$$

We have then the following theorem

Theorem 3.1. *Suppose that the solution of the problem (1.1-1.3) is in $C^{r,3}([\alpha, \beta] \times [0, T])$. For h sufficiently small and $0 \leq m \leq M - 1$,*

$$\|\Gamma^{m+\theta}(\tilde{u}_h^{m+1}, \tilde{u}_h^m)\|_* = \begin{cases} O(h^r + k^2), & \theta = 1/2, \\ O(h^r + k), & \theta > 1/2, \end{cases}$$

and for $i = 0, 1$

$$\left| L_i \left(\frac{\tilde{u}_h^{m+1} - \tilde{u}_h^m}{k} \right) - \frac{g_i(t_{m+1}) - g_i(t_m)}{k} \right| = \begin{cases} O(h^r + k^2), & \theta = 1/2, \\ O(h^r + k), & \theta > 1/2. \end{cases}$$

Proof. Integrating by parts in (1.1), we have

$$(u_t, v) + (a(x)u_x, v_x) - (b(x)u_x, v) - (F(u, x, t), v) = 0, \quad v \in S_h^0.$$

Furthermore, using (2.3), (3.1), and (3.7)-(3.8), it follows that

$$(3.11) \quad ((\tilde{u}_h)_t, v) - (a(x)(\tilde{u}_h)_x, v_x) - (b(x)(\tilde{u}_h)_x, v) - (F(\tilde{u}, x, t), v) \\ = ((\tilde{u}_h)_t - u_t, v) - (F(\tilde{u}, x, t) - F(u, x, t), v).$$

Since

$$\tilde{u}_h(t) - u(t) = p_h(\hat{u}(t)) - \hat{u}(t) + u(\alpha, t)(\phi_0 - v_0) + u(\beta, t)(\phi_1 - v_1),$$

it follows from (2.4), (3.10), that

$$(3.12) \quad \|\tilde{u}_h(t) - u(t)\|_\infty \leq ch^r, \quad \|(\tilde{u}_h)_t(t) - u_t(t)\|_\infty \leq ch^r,$$

from the hypotheses of the theorem, we obtain

$$\|\tilde{u}_h(t_{m+\theta}) - (\theta\tilde{u}_h^{m+1} + (1-\theta)\tilde{u}_h^m)\|_\infty = \begin{cases} O(k^2), & \theta = 1/2, \\ O(k), & \theta > 1/2, \end{cases}$$

$$(3.13) \quad \left\| (\tilde{u}_h)_t(t_{m+\theta}) - \frac{\tilde{u}_h^{m+1} - \tilde{u}_h^m}{k} \right\|_\infty = \begin{cases} O(k^2), & \theta = 1/2, \\ O(k), & \theta > 1/2, \end{cases}$$

thus, for $0 \leq m \leq M-1$,

$$\|\Gamma^{m+\theta}(\tilde{u}_h^{m+1}, \tilde{u}_h^m)\|_* = \begin{cases} O(h^r + k^2), & \theta = 1/2, \\ O(h^r + k), & \theta > 1/2. \end{cases}$$

On the other hand

$$|L_i((\tilde{u}_h)_t(t)) - g'_i(t)| = |L_i((\tilde{u}_h)_t(t) - u_t(t))| \leq ch^r, \quad i = 0, 1,$$

then, for $0 \leq m \leq M$, we have

$$\begin{aligned} & \left| L_i \left(\frac{\tilde{u}_h^{m+1} - \tilde{u}_h^m}{k} \right) - \frac{g_i(t_{m+1}) - g_i(t_m)}{k} \right| \\ & \leq |L_i((\tilde{u}_h)_t(t_{m+\theta}) - g'_i(t_{m+\theta}))| + O(k^2) \\ & = O(h^r + k^2), \quad \theta = 1/2, \quad i = 0, 1, \end{aligned}$$

$$\begin{aligned} & \left| L_i \left(\frac{\tilde{u}_h^{m+1} - \tilde{u}_h^m}{k} \right) - \frac{g_i(t_{m+1}) - g_i(t_m)}{k} \right| \\ & \leq |L_i((\tilde{u}_h)_t(t_{m+\theta}) - g'_i(t_{m+\theta}))| + O(k) \\ & = O(h^r + k), \quad \theta > \frac{1}{2}, \quad i = 0, 1. \end{aligned}$$

The following theorem ensures the existence of a suitable initial condition that satisfies (2.7), which is required for the θ -method.

Theorem 3.2. *Let $V \in S_h$, for h sufficiently small, there exists a unique element $V_* \in S_h$ such that*

$$(3.14) \quad (a(x)(V - V_*)_x, \chi_x) - (b(x)(V - V_*)_x, \chi) = 0, \quad \chi \in S_h^0,$$

$$V_*(\alpha) = \int_{\alpha}^{\beta} K_0(x) V_*(x) dx + g_0(\alpha), \quad V_*(\beta) = \int_{\alpha}^{\beta} K_1(x) V_*(x) dx + g_1(\alpha),$$

Proof. It suffices to show that if W is such that

$$(3.15) \quad (a(x)W_x, \chi_x) - (b(x)W_x, \chi) = 0, \quad \chi \in S_h^0,$$

with

$$(3.16) \quad W(\alpha) = \int_{\alpha}^{\beta} K_0(x) W(x) dx, \quad W(\beta) = \int_{\alpha}^{\beta} K_1(x) W(x) dx,$$

then

$$W = 0.$$

(1.6) implies that

$$(3.17) \quad |W(\alpha)| \leq \mu \|W\|_{\infty}, \quad |W(\beta)| \leq \mu \|W\|_{\infty}.$$

if we set

$$(3.18) \quad \overline{W} = W - W(\alpha)\phi_0 - W(\beta)\phi_1,$$

then $\overline{W} \in S_h^0$, we have by (3.15)

$$(3.19) \quad (a(x)\overline{W}_x, \overline{W}_x) - (b(x)\overline{W}_x, \overline{W}) = 0,$$

this gives

$$\overline{W} = 0,$$

thus

$$W = W(\alpha)\phi_0 + W(\beta)\phi_1.$$

By (3.6), and for h sufficiently small, we obtain

$$\|W\|_{\infty} \leq (1 + \nu) \max(|W(\alpha)|, |W(\beta)|) \leq (1 + \nu)\mu \|W\|_{\infty},$$

if ν is chosen so that $(1 + \nu)\mu < 1$ then $W = 0$. □

V_* can be represented in the form

$$(3.20) \quad V_* = \overline{V} - V_*(\alpha)\phi_0 - V_*(\beta)\phi_1.$$

corollary 3.1. Let u^0 be the initial condition given in (1.3) and $\tilde{u}_h^0 = \tilde{u}_h(t_0)$, for h sufficiently small, if the element $\tilde{u}_{h,*}^0$ is given by

$$(3.21) \quad \left(a(x)(\tilde{u}_h^0 - \tilde{u}_{h,*}^0)_x, \chi_x \right) - \left(b(x)(\tilde{u}_h^0 - \tilde{u}_{h,*}^0)_x, \chi \right) = 0, \quad \chi \in S_h^0,$$

$$\tilde{u}_{h,*}^0(\alpha) = \int_{\alpha}^{\beta} K_0(x) \tilde{u}_{h,*}^0(x) dx + g_0(\alpha), \quad \tilde{u}_{h,*}^0(\beta) = \int_{\alpha}^{\beta} K_1(x) \tilde{u}_{h,*}^0(x) dx + g_1(\alpha),$$

then

$$(3.22) \quad \|\tilde{u}_{h,*}^0 - \tilde{u}_h^0\|_{\infty} \leq Ch^r$$

Proof. We use similar techniques as those in [[5], corollary 3.1].

We define in S_h the following norm

$$\|V\|_{S_h}^2 = (a\overline{V}_x, \overline{V}_x) - (b\overline{V}_x, \overline{V}) + \|V\|_{\infty}^2,$$

for $(V^0, \dots, V^M) \in S_h^{M+1}$

$$\|(V^0, \dots, V^M)\|_{S_h^{M+1}} = \max_{0 \leq m \leq M} \|V^m\|_{\infty},$$

and

$$\begin{aligned} & \left\| \left(V, (\Gamma^{m+\theta})_{m=1}^M, (a_m)_{m=1}^M, (b_m)_{m=1}^M \right) \right\|_{Y_{hk}}^2 \\ &= \|V\|_{S_h}^2 + \sum_{m=1}^M k \|\Gamma^{m+\theta}\|_*^2 + \sum_{m=1}^M k |a_m|^2 + \sum_{m=1}^M k |b_m|^2 \end{aligned}$$

corollary 3.2. Under the hypotheses of theorem 3.1, if $\tilde{u}_h(x, t)$ is the function defined by (3.8) and the initial condition is chosen such that $U^0 = \tilde{u}_{h,*}$, where $\tilde{u}_{h,*} \in S_h$ is defined in corollary (3.1), and $\theta \geq 1/2$, then

$$\|\Phi_{hk}(\tilde{u}_h^0, \dots, \tilde{u}_h^N)\|_{Y_{hk}} = \begin{cases} O(h^r + k^2), & \theta = 1/2, \\ O(h^r + k), & \theta > 1/2. \end{cases}$$

4. Existence and convergence

We first, show a stability result of the θ -method.

Theorem 4.1. *Let R be a fixed real positif and assume that*

*$u \in C^{r,3}([\alpha, \beta] \times [0, T])$, for h and k sufficiently small, $(V^0, \dots, V^M)$
and $(W^0, \dots, W^M) \in B((\tilde{u}_h^0, \dots, \tilde{u}_h^M), R)$ there exists a positive constant*

S such that

$$(4.1) \quad \begin{aligned} & \| (V^0, \dots, V^M) - (W^0, \dots, W^M) \|_{X_{hk}} \\ & \leq S \| \Phi_{hk}(V^0, \dots, V^M) - \Phi_{hk}(W^0, \dots, W^M) \|_{Y_{hk}}. \end{aligned}$$

where $B((\tilde{u}_h^0, \dots, \tilde{u}_h^M), R)$ is the ball of center $(\tilde{u}_h^0, \dots, \tilde{u}_h^M)$ and radius R .

Proof. Noting that for $V \in S_h$, $\bar{V} \in S_h^\circ$ is given by

$$(4.2) \quad \bar{V} = V - V(\alpha)\phi_0 - V(\beta)\phi_1.$$

For $(V^0, \dots, V^M)$ and $(W^0, \dots, W^M) \in B((\tilde{u}_h^0, \dots, \tilde{u}_h^M), R)$, we set
 $R^m = V^m - W^m$, $0 \leq m \leq M$.

$$(4.3) \quad \begin{aligned} & \Gamma^{m+\theta}(V^m, V^{m+1}) \left(\frac{\bar{R}^{m+1} - \bar{R}^m}{k} \right) - \Gamma^{m+\theta}(W^m, W^{m+1}) \left(\frac{\bar{R}^{m+1} - \bar{R}^m}{k} \right) \\ & = \left(\frac{R^{m+1} - R^m}{k}, \frac{\bar{R}^{m+1} - \bar{R}^m}{k} \right) + \left(aR_x^{m+\theta}, \frac{\bar{R}_x^{m+1} - \bar{R}_x^m}{k} \right) \\ & \quad - \left(bR_x^{m+\theta}, \frac{\bar{R}^{m+1} - \bar{R}^m}{k} \right) \\ & - \left(F(V^{m+\theta}, t_{m+\theta}) - F(W^{m+\theta}, t_{m+\theta}), \frac{\bar{R}^{m+1} - \bar{R}^m}{k} \right), \end{aligned}$$

for $0 \leq m \leq M-1$. Then

$$\begin{aligned} & \left\| \Gamma^{m+\theta}(V^m, V^{m+1}) \left(\frac{\bar{R}^{m+1} - \bar{R}^m}{k} \right) - \Gamma^{m+\theta}(W^{m+1}) \left(\frac{\bar{R}^{m+1} - \bar{R}^m}{k} \right) \right\|^2 \\ & \leq \left\| \Gamma^{m+\theta}(V^m, V^{m+1}) - \Gamma^{m+\theta}(W^m, W^{m+1}) \right\|_*^2 \left\| \frac{\bar{R}^{m+1} - \bar{R}^m}{k} \right\|^2, \end{aligned}$$

taking into account (3.1) we get

$$\begin{aligned} & \left(aR_x^{m+\theta}, \frac{\overline{R}_x^{m+1} - \overline{R}_x^m}{k} \right) - \left(bR_x^{m+\theta}, \frac{\overline{R}^{m+1} - \overline{R}^m}{k} \right) \\ &= \frac{1}{k} \left[\theta \left(\left(a\overline{R}_x^{m+1}, \overline{R}_x^{m+1} \right) - \left(b\overline{R}_x^{m+1}, \overline{R}^{m+1} \right) \right) \right. \\ & \quad \left. - (1-\theta) \left(\left(a\overline{R}_x^m, \overline{R}_x^m \right) - \left(b\overline{R}_x^m, \overline{R}^m \right) \right) \right], \end{aligned}$$

hence

$$\begin{aligned} & \frac{1}{k} \left[\theta \left(\left(a\overline{R}_x^{m+1}, \overline{R}_x^{m+1} \right) - \left(b\overline{R}_x^{m+1}, \overline{R}^{m+1} \right) \right) \right. \\ & \quad \left. - (1-\theta) \left(\left(a\overline{R}_x^m, \overline{R}_x^m \right) - \left(b\overline{R}_x^m, \overline{R}^m \right) \right) \right] \\ & \leq \left\| \Gamma^{M+\theta} (V^m, V^{m+1}) - \Gamma^{m+\theta} (W^m, W^{m+1}) \right\|_*^2 \\ & \quad + \left\| F(V^{m+\theta}, t_{m+\theta}) - F(W^{m+\theta}, t_{m+\theta}) \right\|^2 \\ (4.4) \quad & + \frac{1}{2} \left\| \frac{\overline{R}^{m+1} - \overline{R}^m}{k} \right\|^2 - \left(\frac{\overline{R}^{m+1} - \overline{R}^m}{k}, \frac{R^{m+1} - R^m}{k} \right), \end{aligned}$$

$$\begin{aligned} & \text{Moreover} \\ & \left(\frac{\overline{R}^{m+1} - \overline{R}^m}{k}, \frac{R^{m+1} - R^m}{k} \right) \\ (4.5) \quad &= \left\| \frac{R^{m+1} - R^m}{k} \right\|^2 - \left(\frac{R^{m+1} - R^m}{k}, \frac{R^{m+1}(\alpha) - R^m(\alpha)}{k} \emptyset_0 \right) \\ & \quad - \left(\frac{R^{m+1} - R^m}{k}, \frac{R^{m+1}(\beta) - R^m(\beta)}{k} \emptyset_1 \right), \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{\overline{R}^{m+1} - \overline{R}^m}{k} \right\|^2 &= \left\| \frac{R^{m+1} - R^m}{k} \right\|^2 \\ & \quad - 2 \left(\frac{R^{m+1} - R^m}{k}, \frac{R^{m+1}(\alpha) - R^m(\alpha)}{k} \emptyset_0 \right) \\ & \quad - 2 \left(\frac{R^{m+1} - R^m}{k}, \frac{R^{m+1}(\beta) - R^m(\beta)}{k} \emptyset_1 \right) \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{R^{m+1}(\alpha) - R^m(\alpha)}{k} \right|^2 \|\emptyset_0\|^2 + \left| \frac{R^{m+1}(\beta) - R^m(\beta)}{k} \right|^2 \|\emptyset_1\|^2 \\
 & + 2 \left(\frac{R^{m+1}(\alpha) - R^m(\alpha)}{k} \emptyset_0, \frac{R^{m+1}(\beta) - R^m(\beta)}{k} \emptyset_1 \right).
 \end{aligned}$$

It follows by replacing all these inequalities in (4.4) that

$$\begin{aligned}
 & \frac{1}{k} \left[\theta \left((a\bar{R}_x^{m+1}, \bar{R}_x^{m+1}) - (bR_x^{m+1}, R_x^{m+1}) \right) \right. \\
 & \quad \left. - (1-\theta) \left((a\bar{R}_x^m, \bar{R}_x^m) - (b\bar{R}_x^m, \bar{R}_x^m) \right) \right] \\
 & \leq \left\| \Gamma^{m+\theta}(V^m, V^{m+1}) - \Gamma^{m+\theta}(W^m, W^{m+1}) \right\|_*^2 \\
 & \quad + \left\| F(V^{m+\theta}, t_{m+\theta}) - F(W^{m+\theta}, t_{m+\theta}) \right\|^2 \\
 & \quad - \frac{1}{2} \left\| \frac{R^{m+1} - R^m}{k} \right\|^2 + \frac{1}{2} \left| \frac{R^{m+1}(\alpha) - R^m(\alpha)}{k} \right|^2 \|\emptyset_0\|^2 \\
 & \quad \quad \quad + \frac{1}{2} \left| \frac{R^{m+1}(\beta) - R^m(\beta)}{k} \right|^2 \|\emptyset_1\|^2 \\
 (4.6) \quad & + \left| \frac{R^{m+1}(\alpha) - R^m(\alpha)}{k} \right| \left| \frac{R^{m+1}(\beta) - R^m(\beta)}{k} \right| (|\emptyset_0|, |\emptyset_1|),
 \end{aligned}$$

for h sufficiently small, the last three terms of (4.6) are bounded by

$$\begin{aligned}
 & \frac{1}{4} \left| \frac{R^{m+1}(\alpha) - R^m(\alpha)}{k} \right|^2 (1 + \|\emptyset_0\|^2 - \|\emptyset_1\|^2 + \nu) \\
 (4.7) \quad & + \frac{1}{4} \left| \frac{R^{m+1}(\beta) - R^m(\beta)}{k} \right|^2 (1 + \|\emptyset_1\|^2 - \|\emptyset_0\|^2 + \nu),
 \end{aligned}$$

combining (1.6), the Cauchy-Schwartz inequality, and the fact that $(a+b)^2 \leq (1+\delta)a^2 + (1+\frac{1}{\delta})b^2$, for any real numbers a, b and for any $\delta > 0$ we get

$$\begin{aligned}
 & \left| \frac{R^{m+1}(\alpha) - R^m(\alpha)}{k} \right|^2 \leq (1 + \frac{\delta}{4})\mu^2 \left\| \frac{R^{m+1} - R^m}{k} \right\|^2 + (1 + \frac{4}{\delta}) \left| L_0 \left(\frac{R^{m+1} - R^m}{k} \right) \right|^2 \\
 & \left| \frac{R^{m+1}(\beta) - R^m(\beta)}{k} \right|^2 \leq (1 + \frac{\delta}{4})\mu^2 \left\| \frac{R^{m+1} - R^m}{k} \right\|^2 + (1 + \frac{4}{\delta}) \left| L_1 \left(\frac{R^{m+1} - R^m}{k} \right) \right|^2, \\
 (4.8)
 \end{aligned}$$

If we substitute (4.7)-(4.8) into (4.6), we obtain

$$\begin{aligned}
& \theta \left((a\overline{R}_x^{m+1}, \overline{R}_x^{m+1}) - (b\overline{R}_x^{m+1}, \overline{R}^{m+1}) \right) - (1-\theta) \left((a\overline{R}_x^m, \overline{R}_x^m) - (b\overline{R}_x^m, \overline{R}^m) \right) \\
& \leq k \left\| \Gamma^{m+\theta}(V^m, V^{m+1}) - \Gamma^{m+\theta}(W^m, W^{m+1}) \right\|_*^2 \\
& \quad + k \left\| F(V^{m+\theta}, t_{m+\theta}) - F(W^{m+\theta}, t_{m+\theta}) \right\|^2 \\
& \quad + k \left(\frac{1 + \|\phi_0\|^2 - \|\phi_1\|^2 + \nu}{4} \right) \left(1 + \frac{4}{\delta} \right) \left| L_0 \left(\frac{R^{m+1} - R^m}{k} \right) \right|^2 \\
& \quad + k \left(\frac{1 + \|\phi_1\|^2 - \|\phi_0\|^2 + \nu}{4} \right) \left(1 + \frac{4}{\delta} \right) \left| L_1 \left(\frac{R^{m+1} - R^m}{k} \right) \right|^2 \\
(4.9) \quad & - \frac{k}{2} \left[1 - (1+\nu)\mu^2(\beta - \alpha) \left(1 + \frac{\delta}{4} \right) \right] \left\| \frac{R^{m+1} - R^m}{k} \right\|^2,
\end{aligned}$$

$0 \leq m \leq M-1$. If ν and δ are chosen such that $(1+\nu)\mu^2(1+\frac{\delta}{4}) < 1$ and using the fact that the function F is Lipschitz continuous (4.8) takes the form

$$\begin{aligned}
& \frac{\theta}{1-\theta} \left((a\overline{R}_x^{m+1}, \overline{R}_x^{m+1}) - (b\overline{R}_x^{m+1}, \overline{R}^{m+1}) \right) - \left((a\overline{R}_x^m, \overline{R}_x^m) - (b\overline{R}_x^m, \overline{R}^m) \right) \\
& \leq \frac{k}{1-\theta} \left\| \Gamma^{m+\theta}(V^m, V^{m+1}) - \Gamma^{m+\theta}(W^m, W^{m+1}) \right\|_*^2 \\
& \quad + \frac{Ck}{1-\theta} (\|R^{m+1}\|^2 + \|R^m\|^2) \\
(4.10) \quad & + \frac{Ck}{1-\theta} \left(\left| L_0 \left(\frac{R^{m+1} - R^m}{k} \right) \right|^2 + \left| L_1 \left(\frac{R^{m+1} - R^m}{k} \right) \right|^2 \right),
\end{aligned}$$

$0 \leq m \leq M-1$, summing on m we have

$$\begin{aligned}
& (a\overline{R}_x^m, \overline{R}_x^m) - (b\overline{R}_x^m, \overline{R}^m) - \left(\frac{1-\theta}{\theta} \right)^m \left((a\overline{R}_x^0, \overline{R}^0) - (b\overline{R}_x^0, \overline{R}^0) \right) \\
& \leq C \left(\sum_{n=1}^m \frac{(1-\theta)^{m-n}}{\theta^{m-n+1}} k \left\| \Gamma^{n-1+\theta}(V^m, V^{m-1}) - \Gamma^{n-1+\theta}(W^m, W^{m-1}) \right\|_*^2 \right. \\
& \quad \left. + \sum_{n=1}^m \frac{(1-\theta)^{m-n}}{\theta^{m-n+1}} k \left[\left| L_0 \left(\frac{R^n - R^{n-1}}{k} \right) \right|^2 + \left| L_1 \left(\frac{R^n - R^{n-1}}{k} \right) \right|^2 \right] \right)
\end{aligned}$$

$$(4.11) + \sum_{n=0}^m \frac{(1-\theta)^{m-n-1}}{\theta^{m-n+1}} k \|R^n\|^2 + \frac{(1-\theta)^{m-1}}{\theta^m} k \|R^0\|^2 + \frac{k}{\theta} \|R^m\|^2$$

$1 \leq m \leq M$.

Since $1/2 \leq \theta \leq 1$ then $\frac{1-\theta}{\theta} \leq 1$.

On the other hand, using Sobolev's inequality we get

$$(4.12) \quad \|R^m\|_\infty \leq C \left(\|\bar{R}_x^m\| + \|\bar{R}^m\| + |L_0(R^m)| + |L_1(R^m)| \right),$$

and

$$(4.13) \quad |L_i(R^m)|^2 \leq 2T \sum_{j=1}^m k \left| L_i \left(\frac{R^j - R^{j-1}}{k} \right) \right|^2 + |L_i(R^0)|^2, \quad i = 0, 1,$$

from (1.4)

$$(4.14) \quad a_0 \|\bar{R}_x^m\|^2 + 1/2b_0 \|\bar{R}^m\|^2 \leq (a\bar{R}_x^m, \bar{R}_x^m) - (b\bar{R}_x^m, \bar{R}^m) \leq a_1 \|\bar{R}_x^m\|^2 + 1/2b_1 \|\bar{R}^m\|^2,$$

thus

$$\begin{aligned} \|R^m\|_\infty^2 &\leq C \left((a\bar{R}_x^0, \bar{R}_x^0) - (b\bar{R}_x^0, \bar{R}^0) + |L_0(R^0)|^2 + |L_1(R^0)|^2 \right. \\ &\quad \left. + \sum_{n=1}^m k \left\| \Gamma^{n-1+\theta}(V^n, V^{n-1}) - \Gamma^{n-\theta}(W^n, W^{n-1}) \right\|_*^2 + k \|R^0\|^2 \right. \\ &\quad \left. + \sum_{n=1}^m k \left[\left| L_0 \left(\frac{R^n - R^{n-1}}{k} \right) \right|^2 + \left| L_1 \left(\frac{R^n - R^{n-1}}{k} \right) \right|^2 \right] + \sum_{m=1}^n k \|R^n\|_\infty^2 \right), \end{aligned}$$

to conclude, we use Gronwal's inequality (discrete form), which gives, for k sufficiently small,

$$\begin{aligned} \|R^m\|_\infty^2 &\leq C \left(\|R^0\|_{S_h}^2 + \sum_{n=1}^m k \left\| \Gamma^{n-\theta}(V^n, V^{n-1}) - \Gamma^{n-\theta}(W^n, W^{n-1}) \right\|_*^2 \right. \\ &\quad \left. + \sum_{n=1}^m \left[\left| L_0 \left(\frac{R^n - R^{n-1}}{k} \right) \right|^2 + \left| L_1 \left(\frac{R^n - R^{n-1}}{k} \right) \right|^2 \right] \right), \end{aligned}$$

for $0 \leq m \leq M$, thus

$$\begin{aligned} &\|(V^0, \dots, V^M) - (W^0, \dots, W^M)\|_{X_{hk}} \\ &\leq S \|\Phi_{hk}(V^0, \dots, V^M) - \Phi_{hk}(W^0, \dots, W^M)\|_{Y_{hk}}, \quad \text{which completes the proof.} \end{aligned}$$

Analogous to theorem 5.1 of [5], the following result holds:

Theorem 4.2. Let be $(\tilde{u}_h^0, \dots, \tilde{u}_h^M)$ defined as in (3.9) and Φ_{hk} the operator defined as in (2.5) with $U^0 = \tilde{u}_{h,*}^0$ if R is a fixed positif real, for h and k sufficiently small, (2.8) possesses a unique solution $(U^0, \dots, U^M) \in B((\tilde{u}_h^0, \dots, \tilde{u}_h^M), R)$ that satisfies

$$\|(U^0, \dots, U^M) - (\tilde{u}_h^0, \dots, \tilde{u}_h^M)\|_{Y_{hk}} = \begin{cases} O(h^r + k^2), & \theta = 1/2, \\ O(h^r + k), & \theta > 1/2. \end{cases}$$

The following corollary is a consequence of the above theorem.

corollary 4.1. For $\theta \geq 1/2$, under the hypotheses of theorem 4.2, if $(U^0, \dots, U^M)$ is the solution of (2.8), and consider

$$u^m(x) = u(x, t_m), \quad 0 \leq m \leq M,$$

where u is the solution of (1.1-1.3), then

$$\|U^m - u^m\|_\infty = \begin{cases} O(h^r + k^2), & \theta = 1/2, \\ O(h^r + k), & \theta > 1/2, \quad 0 \leq m \leq M. \end{cases}$$

5. Numerical results

To observe the behavior of the θ - method, it was tested on the following problem [1].

$$u_t - u_{xx} = -e^{-t} \left\{ x(x-1) + \frac{\delta}{6(1+\delta)} + 2 \right\}, \quad x \in (0, 1), \quad 0 < t \leq T,$$

$$u(0, t) = u(1, t) = -\delta \int_0^1 u(x, t) dx, \quad t > 0,$$

$$u(x, 0) = x(x-1) + \frac{\delta}{6(1+\delta)}, \quad x \in [0, 1].$$

Where $|\delta| < 1$ and the exact solution, $u = u(x, t)$, is given by

$$u(x, t) = e^{-t} \left(x(x-1) + \frac{\delta}{6(1+\delta)} \right).$$

Below are tables, representing sample calculations using three values for the parameter θ of the described methods. The first and second column in the table represents the mesh parameter M and the time steps N . The third column shows the error measured in maximum norm on the $M \times N$ mesh, covering $[0, 1] \times [0, T]$ and the last column shows the CPU-time used, in seconds, for the computation.

Table 5.1: Crank-Nicholson method, $\theta = \frac{1}{2}$

M	N	error	CPU-time
2	2	4.04e-04	8.61e-01
4	4	2.19e-04	9.1e-01
8	8	7.07e-05	9.8e00
16	16	2.37e-05	5.22e01
32	32	3.55e-06	6.0e02
64	64	1.87e-06	1.2e03
128	128	6.35e-07	3.9e03

Table 5.2: $\theta = 0$

M	N	error	CPU-time
2	2	7.21e-04	9.21e-01
4	4	9.8e-04	1.1e00
8	8	8.04e-05	1.03e01
16	16	5.87e-05	6.34e01
32	32	2.45e-05	7.21e02
64	64	7.25e-06	2.4e03
128	128	3.23e-06	5.8e03

Table 5.3: $\theta = 1$

M	N	error	CPU-time
2	2	8.4e-04	1.6e00
4	4	5.14e-04	2.11e00
8	8	3.7e-04	13.98e00
16	16	8.27e-05	5.22e01
32	32	3.5e-05	8.59e02
64	64	8.67e-06	2.52e03
128	128	4.86e-06	7.75e03

Comparing the above three tables, we find that the Crank-Nicholson method is, unsurprisingly, more efficient than the others.

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