PARABOLIC PERTURBATION IN THE FAMILY $z \mapsto 1 + 1/wz^d$

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Abstract

Consider the family of rational maps $\mathcal{F}_d = \{z \mapsto f_w(z) = 1 + \frac{1}{wz^d} : w \in \mathbb{C}\setminus\{0\}\}$ ($d \in \mathbb{N}, d \geq 2$), and the hyperbolic component $A_1 = \{w : f_w \text{ has an attracting fixed point} \}$. We prove that if $w_0 \in \partial A_1$ is a parabolic parameter with corresponding multiplier a primitive $q$–th root of unity, $q \geq 2$, then there exists a hyperbolic component $W_q$, attached to $A_1$ at the point $w_0$, which contains $w$–values for which $f_w$ has an attracting periodic cycle of period $q$.

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1. Introduction

For any $d \in \mathbb{N}, d \geq 2$, the family $\mathcal{F}_d = \{ z \mapsto 1 + \frac{1}{w z^d} : w \in \mathbb{C} \setminus \{0\} \}$ is a normal form for the set of rational maps of degree $d$ which have exactly two critical points, one of which maps onto the other under one iteration. These families have been considered in [3], [4] (for the special case $d = 2$), and in [1](for any $d$).

It is well known that a rational map $f$ is hyperbolic if and only if all critical points of $f$ tend to attracting cycles under iteration. Since the members of the family $\mathcal{F}_d$ have only one forward orbit of their critical points, $f_w$ is hyperbolic if and only if $f_w$ has an attracting periodic orbit. The connected components of the parametric set $\mathcal{H}_d = \{ w : f_w(z) = 1 + 1/w z^d \text{ has an attracting periodic orbit} \}$ are called the hyperbolic components of the family $\mathcal{F}_d$.

Following the ideas used in [2], we can prove the following one.

**Theorem** If $f_{w_0}$ has a fixed point $z_0$ such that the multiplier $\lambda_0 = f'_{w_0}(z_0)$ is a primitive $q$–th root of unity, $q \geq 2$, then there exists a hyperbolic component $W_q$, which contains $w$–values for which $f_w$ has an attracting periodic cycle of period $q$, with $w_0 \in \partial W_q$.

In Section 2 we prove the Theorem.

2. Proof of Theorem

It is clear that for any $u \in \mathbb{C} \setminus \{0, -d\}$,

$\begin{align*}
f_w \text{ has a fixed point of multiplier } u & \iff w = \frac{d}{u} \left(1 + \frac{1}{u^d}\right)^{d+1}.
\end{align*}$

$\begin{align*}
f_w \text{ has a fixed point of multiplier } u & \iff w = -\frac{d}{u} \left(1 + \frac{1}{u^d}\right)^{d+1}.
\end{align*}$

In fact, $z(u) = \frac{d}{d + u}$ is the fixed point of multiplier $u$.

Let $g_u(z) := \frac{d}{d + u} \left(1 + \frac{1}{u^d}\right)^{d+1} (z)$, that is,

$\begin{align*}
g_u(z) = 1 - \frac{d^d u}{(d + u)^{d+1} z^d}, \quad u \in \mathbb{C} \setminus \{0, -d\}.
\end{align*}$

Therefore, $g_u$ has at $z(u) = \frac{d}{d + u}$ a fixed point of multiplier $u$. Now, we will make an analytic conjugation:
Let \( M_u(z) := z - z(u) \), and consider \( h_u := M_u \circ g_u \circ M_u^{-1} \).

For any \( u \in \mathbb{C} \setminus \{0, -d\} \), \( h_u \) is a rational map analytically conjugate to \( f_w \) (where \( w = -\frac{d}{u} \left( 1 + \frac{u}{d} \right)^{d+1} \)), and which has at zero a fixed point of multiplier \( u \).

Explicitly,
\[
h_u(z) = \frac{u}{d+u} \cdot \frac{(z(d+u)+d)^{-d}}{(z(d+u)+d)^d}, \quad u \in \mathbb{C} \setminus \{0, -d\}.
\]

Note that, for any \( q \in \mathbb{N} \), \( h_u^q(z) = u^q z \cdot \Phi_{u,q}(z) \), where \( \Phi_{u,q} \) is a rational map with \( \Phi_{u,q}(0) = 1 \). Hence, in a neighbourhood of \( z = 0 \) we have:
\[
h_u^q(z) = u^q z + a_2 z^2 + \ldots.
\]

In what follows, \( u_0 \) denotes a primitive \( q \)-th root of unity, \( q \geq 2 \) (that is, \( u_0^q = 1 \), and \( u_0^k \neq 1 \), for all \( 1 \leq k \leq q - 1 \)). In order to prove the above theorem, we show the following results:

**Lemma 1:** \( h_{u_0}^q \) has at \( 0 \) a fixed point of multiplicity \( q + 1 \).

**Proof:** Since \( h_{u_0} \) has at zero a fixed point of multiplier a primitive \( q \)-th root of unity, we have that in a neighbourhood of zero,
\[
h_{u_0}^q(z) = z + a z^{kq+1} + \ldots \text{ where } a \neq 0, \text{ and } k \in \mathbb{N}.
\]

From the fact that \( h_{u_0} \) has only one forward orbit of critical points, \( k = 1 \).

Therefore, \( h_{u_0}^q(z) = z + a z^{q+1} + \ldots. \)

Next, we will show that for \( u \) near to \( u_0 \), the \( (q + 1) \)-fold fixed point zero of \( h_{u_0}^q \) will split up into \( (q+1) \) simple fixed points of \( h_u^q \), which are: \( 0 \), and \( \{ z_1(u), z_2(u), \ldots, z_q(u) \} \); the latter forms a periodic orbit of period \( q \) of \( h_u \).

**Lemma 2:** There exist \( \varepsilon > 0 \) and \( r > 0 \) such that for each \( u \in \mathbb{C} \) with \( 0 < |u - u_0| < r \), the rational map \( h_u^q \) has precisely \( q \) fixed points in the punctured disc \( 0 < |z| < \varepsilon \). Furthermore, these \( q \) points form a cycle of length \( q \) for \( h_u \).
Proof: Since the zeros of an analytic function (not identically zero) are isolated, there exist $\varepsilon, \ 0 < \varepsilon < \frac{d-1}{d+2}$, such that:

$$h^k_{u_0}(z) - z \neq 0 \quad \text{for} \quad 0 < |z| < \varepsilon', \quad \text{and} \quad 1 \leq k \leq q,$$

where $\varepsilon' := \frac{2}{d-1}[(2d-1)^d - d^d] \cdot \varepsilon$. (Note that $\varepsilon < \varepsilon'$).

Let $\gamma_{\varepsilon} = \{ z \in \mathbb{C} : |z| = \varepsilon \}$, $\gamma_{\varepsilon'} = \{ z \in \mathbb{C} : |z| = \varepsilon' \}$, and

$$\alpha := \min_{1 \leq k \leq q} \{ |h^k_{u_0}(z) - z| : z \in \gamma_{\varepsilon} \cup \gamma_{\varepsilon'} \} > 0.$$

It is clear that there exists $r, \ 0 < r < 1$, such that:

$$|h^k_u(z) - z| \geq \frac{\alpha}{2}, \ \text{for all} \ |u - u_0| < r, \ z \in \gamma_{\varepsilon} \cup \gamma_{\varepsilon'}, \ \text{and} \ 1 \leq k \leq q.$$

From the Argument Principle, the number $N_{k,\varepsilon}(u)$ (resp. $N_{k,\varepsilon'}(u)$) of fixed points of $h^k_u$ in the disk $|z| < \varepsilon$ (Resp. $|z| < \varepsilon'$) for $|u - u_0| < r$ and $1 \leq k \leq q$, is given by:

$$N_{k,\varepsilon}(u) = \frac{1}{2\pi i} \oint_{|z| = \varepsilon} \frac{(h^k_u)'(z) - 1}{h^k_u(z) - z} \, dz$$

(resp. $N_{k,\varepsilon'}(u) = \frac{1}{2\pi i} \oint_{|z| = \varepsilon'} \frac{(h^k_u)'(z) - 1}{h^k_u(z) - z} \, dz$)

From above we conclude that $u \mapsto N_{k,\varepsilon}(u)$, and $u \mapsto N_{k,\varepsilon'}(u)$, are continuous, and hence are constant since they are integer-valued.

Therefore,

$$N_{k,\varepsilon}(u) = N_{k,\varepsilon}(u_0) \quad \text{and} \quad N_{k,\varepsilon'}(u) = N_{k,\varepsilon'}(u_0)$$

for $|u - u_0| < r$ and $1 \leq k \leq q$.

Hence, $|u - u_0| < r \implies$

$$\left\{ \begin{array}{ll}
N_{k,\varepsilon}(u) = N_{k,\varepsilon'}(u) = 1 & , \quad \text{if} \quad 1 \leq k \leq (q - 1) \\
N_{k,\varepsilon}(u) = N_{k,\varepsilon'}(u) = (q + 1) & , \quad \text{if} \quad k = q.
\end{array} \right.$$ 

We conclude that 0 is the unique fixed point of $h^k_u(1 \leq k \leq (q - 1))$ in the disk $|z| < \varepsilon'$. On the other hand, for $0 < |u - u_0| < r$, $h^q_u$ has at zero a simple fixed point, and has other $q$ fixed points in the punctured disk $0 < |z| < \varepsilon$. Note that for $|u - u_0| < r$, $h^q_u$ has no fixed points in $\varepsilon < |z| < \varepsilon'$, because $N_{q,\varepsilon}(u) = N_{q,\varepsilon'}(u)$.

Furthermore, using the facts that $\varepsilon < \frac{d}{d+2}$, $r < 1$, a simple calculation shows that:

$$\forall \ u \in \{ u : |u - u_0| < r \} , \quad |z| < \varepsilon \ \Rightarrow \ |h^k_u(z)| < \varepsilon'.$$
Hence, if $z_1(u)$ denotes one of the fixed points of $h_u^q$ with $0 < |z_1(u)| < \varepsilon$, then $z_j(u) = h_u^j(z_1(u))$, for $0 \leq j \leq (q - 1)$, are the $q$ fixed points of $h_u^q$ in the punctured disk $0 < |z| < \varepsilon$ (they are clearly different pairwise).

Therefore, $\{z_1(u), h_u(z_1(u)), \ldots, h_u^{q-1}(z_1(u))\}$ are the $q$ fixed points of $h_u^q$ in the punctured disk $0 < |z| < \varepsilon$, and they form a cycle of length $q$ of $h_u$, for $u \in \{u : 0 < |u - u_0| < r\}$.

For $u \in \{u : 0 < |u - u_0| < r\}$, $\lambda(u)$ denotes the multiplier of the periodic cycle of period $q$ of $h_u$, contained in the punctured disk $0 < |z| < \varepsilon$.

**Lemma 3:** $u \mapsto \lambda(u)$ is an analytic function in the disk $\{u : |u - u_0| < r\}$. Furthermore, $\lambda(u_0) = 1$.

**Proof:** For $0 < |u - u_0| < r$, let $\{z_1(u), z_2(u), \ldots, z_q(u)\}$ be the periodic cycle of period $q$ of $h_u$, contained in the punctured disk $0 < |z| < \varepsilon$. Furthermore, for $u = u_0$, let $z_1(u_0) = z_2(u_0) = \ldots = z_q(u_0) = 0$.

Consider the polynomial $P_u(z) = \prod_{j=1}^q (z - z_j(u))$.

We know that $P_u(z) = z^q + a_{q-1}(u)z^{q-1} + \ldots + a_1(u)z + a_0(u)$, where,

$$a_{q-k}(u) = (-1)^k \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq q} z_{j_1}(u)z_{j_2}(u) \ldots z_{j_k}(u)$$

are the elementary symmetric functions in $z_1(u), z_2(u), \ldots, z_q(u)$.

Consider the polynomials:

$$\sigma_k(u) = \sum_{j=1}^q (z_j(u))^k, \quad k = 1, 2, \ldots, q$$

A calculation shows that each elementary symmetric function can be written as a polynomial in $\sigma_1(u), \sigma_2(u), \ldots, \sigma_q(u)$. Indeed, we have that:

$$a_{q-1}(u) = -\sigma_1(u) \quad a_{q-2}(u) = \frac{1}{2}[\sigma_1(u)^2 - \sigma_2(u)]$$

$$a_{q-3}(u) = \frac{1}{6}[3\sigma_1(u)\sigma_2(u) - 2\sigma_3(u) - (\sigma_1(u))^3]$$

and, so on.
On the other hand, by the Residue Theorem we have that for 
$0 < |u - u_0| < r$, and $1 \leq k \leq q$,

$$
\sigma_k(u) = \frac{1}{2\pi i} \oint_{|z| = \varepsilon} z^k \frac{(h_u^q)'(z) - 1}{h_u^q(z) - z} \, dz
$$

Note that the above formula holds also for $u = u_0$.

Hence, by the Leibniz’s rule we conclude that :

$$
\forall \ k \in \{1, 2, \ldots, q\}, \ u \mapsto \sigma_k(u)
$$

is holomorphic in the disk $|u - u_0| < r$.

Therefore, $a_0(u), a_1(u), \ldots, a_{q-1}(u)$ are holomorphic functions in
the disk $|u - u_0| < r$.

For the multiplier, we have :

$$
\lambda(u) = (h_u^q)'(z_1(u)) = \prod_{j=1}^{q} h_u'(z_j(u)) = \prod_{j=1}^{q} \frac{ud^{d+1}}{[z_j(u)(d + u) + d]^{d+1}}
$$

Hence,

$$
\lambda(u) = \frac{d^{(d+1)q}u^q}{[\prod_{j=1}^{q}(z_j(u)(d + u) + d)]^{d+1}}, \ \forall \ |u - u_0| < r
$$

Since, $\forall \ u \in \{u : |u - u_0| < r\}$, $\prod_{j=1}^{q}(z_j(u)(d + u) + d) = (-d + u)^q \prod_{j=1}^{q}(\frac{d}{d + u} - z_j(u)) = (-d + u)^q P_u\left(\frac{-d}{d + u}\right)$,

we conclude that $u \mapsto \lambda(u)$ is analytic in $|u - u_0| < r$. Finally, is clear that $\lambda(u_0) = 1$.

**Proof of Theorem:** If $f_{w_0}(z) = 1 + 1/w_0 z^d$ has a fixed point $z_0$ with corresponding multiplier $u_0 = f'_{w_0}(z_0)$ a primitive $q$–th root of unity, $q \geq 2$, then $h_{u_0}$ has at zero a fixed point of multiplier $u_0$.

By lemma 2, there exists $r > 0$ such that for each $u \in \mathbb{C}$ with $0 < |u - u_0| < r$, the rational map $h_u$ has a periodic orbit $\{z_1(u), z_2(u), \ldots, z_q(u)\}$ of period $q$. Furthermore, by lemma 3 the multiplier $\lambda(u)$ of that periodic orbit, is an analytic function
Parabolic perturbation in the family \( z \mapsto 1 + 1/wz^d \) in the disk \( B(u_0, r) := \{ u : |u - u_0| < r \} \), where \( \lambda(u_0) = 1 \). \( \lambda \) is clearly non-constant, and therefore is open. Then we conclude that there exists a hyperbolic component of period \( q \), \( W_q \), such that \( w_0 \in \partial W_q \).

References


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