Abstract

Let $A \cong kQ/I$ be a basic and connected finite dimension algebra over closed field $k$. In this note show that in case $B = A[M]$ is a tame one-point extension of a tame concealed algebra $A$ by an indecomposable module $M$, then the trivial extension $T(B) = B \propto DB$ is tame if and only if the module $M$ is regular.
1. Introduction.

Throughout this paper, $k$ denotes an algebraically closed field. By algebra $A$, we mean always a basic, connected and finite dimensional algebra over $k$ (associative with 1). We denote by $\text{mod} A$ the category of finitely generated right $A$-modules, and $\mathcal{D}^b(A)$ the derived category of bounded complexes over the abelian category $\text{mod} A$ (see [H]).

The concept of repetitive algebra was introduced by Hughes - Waschbush ([HW]) in 1983, where their main interest was to obtain the classification of the finite representation self-injective algebras. In section 2 we recall some known facts about repetitive algebra. In this note, we will use the properties of repetitive categories to study the representation type of the trivial extension $T(B) = B \bowtie DB$, where $B$ is a one-point extension of a tame concealed algebra by an indecomposable module.

In section 3 we establish our main theorem on the representation type of the trivial extension $T(B)$. For that purpose, we prove that there exist a strong relation between the trivial extension $T(B)$ and the class of clannish algebras introduced by Crawley-Boevey in [C-B]. As a consequence of our main theorem we show that all tree algebra with non-negative Euler form $\chi_A$ of corank $\chi_A \leq 2$, have trivial extension of tame representation type.

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2. Preliminaries.

We recall that a quiver $Q = (Q_0, Q_1)$ is an oriented graph, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows. The ordinary quiver associated to an algebra $A$ will be denote by $Q_A$. The $k$-algebra $A$ will be called triangular when $Q_A$ has no oriented cycles. For each vertex $i$ of $Q_A$, we shall denote by $e_i$ the corresponding primitive idempotent of $A$, and by $S_i$ the corresponding simple $A$-module. We denote $P_i$ (respectively $I_i$) the projective cover (respectively, the injective envelope) of $S_i$. A bound quiver algebra $A \cong kQ/I$ will sometimes be considered as a $k$-category.
Let $\mathcal{H}$ be a Krull-Schmidt category. By definition, the quiver $\Gamma(\mathcal{H})$ of $\mathcal{H}$ has as vertices the isomorphism classes $[M]$ of indecomposable objects $M \in \mathcal{H}$, and there are many arrows $[M] \rightarrow [N]$ as the dimension of the space of irreducible maps from $M$ to $N$ in $\mathcal{H}$ (see VII.1 [ARS]). If $\mathcal{H} = \text{mod}\, A$ or $D^b(A)$, then $\Gamma(\mathcal{H})$ is a translation quiver (see 2.1 in [R]). The quiver $\Gamma(\text{mod}\, A)$, or $\Gamma_A$, is called the Auslander-Reiten quiver of $A$. A translation quiver $\Gamma$ is called a tube (see V III. 4 in [ARS]), if it contains cyclic paths and its topological realization is $|\Gamma| = S^1 \times \mathbb{R}^+_0$ (where $S^1$ is the unit circle and $\mathbb{R}^+_0$ is the set of non-negative real numbers). A $k$-category $A$ is called $\tilde{A}$-free whenever it contains no full sub category $B \cong kQ$ where the underlying graph of $Q$ is $\mathbb{A}_n$, for some $n$.

For the basic definitions and results of tilting theory, we refer the reader to [A1]. Two finite-dimensional $k$-algebras $A$ and $B$ are called tilting-cotilting equivalent, if there exist a sequence of algebras $A = A_0, A_1, ..., A_{m+1} = B$ and a sequence of modules $T^i_A$ ($0 \leq i \leq m$) such that $A_{i+1} = \text{End}T^i_A$, and $T^i_A$ is either a tilting or cotilting module (see [A1]).

The one-point extension (respectively, coextension) of an algebra $A$ by an $A$-module $M$ will be denoted by $A[M]$ (respectively, $[M]A$). Let $A$ be a triangular algebra and $i$ a sink in $Q_A$. The reflection $S^+_i A$ (see [HW]), of $A$ is defined as the quotient of the one-point extension $A[M]$ by the bilateral ideal generated by $e_i$. Dually, starting with a source $j$, we define the reflection $S^-_j A$.

By a polynomial-growth critical algebra, shortly pg-critical algebra (see 3 in [NS]) we mean an algebra $A$ satisfying the following conditions:

1) $A$ or $A^{op}$ is of one of the following form:

\[
C[M] = \begin{bmatrix} k & M \\ 0 & C \end{bmatrix}, \quad C[N,t] = \begin{bmatrix} k & k & \ldots & k & k & N \\ k & k & \ldots & k & k & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ k & k & \ldots & k & 0 \\ 0 & k & \ldots & k & 0 \\ 0 & 0 & \ldots & 0 & C \end{bmatrix}
\]
where $C$ is a representation infinite tilted algebra of type $\tilde{D}_n$ with $(4 \leq n)$, with a complete slice in the preinjective component, and $M$ (respectively, $N$) is an indecomposable regular $C$-module of regular length 2 (respectively, regular length 1) lying in a tube $T$ in $\Gamma_C$ having $n - 2$ rays, and $t + 1$ ($2 \leq t$) is the number of objects in $C[N,t]$ which are not in $C$.

2) Every proper convex sub category of $A$ is of polynomial growth.

In particular, we say that the algebra $A$ is 2-tubular if $A = \tilde{D}_n[M]$, where $M \in \text{ind}\tilde{D}_n$ is regular indecomposable of length 2 lying in a tube $T$ in $\Gamma_{\tilde{D}_n}$ having $n - 2$ rays.

**Proposition 2.1** *(1.4 in [P1]).* A pg-critical algebra $A$ is derived-equivalent to an algebra given by the following quiver:

![Figure 1](image)

With the commutative relations, indicated by dotted edges.

Let $A$ be a finite-dimensional $k$-algebra, and $D = Hom_k(-, k)$ denote the standard duality on $\text{mod}A$. The repetitive algebra $\tilde{A}$ (see [HW]) of $A$ is the self-injective, locally finite-dimensional algebra without identity, defined by:

$$
\tilde{A} = \begin{pmatrix}
\ddots & & & & 0 \\
& A & & & \\
& DA & A & & \\
& DA & A & \ddots & \\
0 & & & & \ddots
\end{pmatrix}
$$
where matrices have only finitely many non-zero entries, addition is the usual addition of matrices, and multiplication is induced from the canonical bimodule structure of $DA = \text{Hom}_k(A, k)$ and the zero map $DA \otimes DA \to 0$.

It was proved in [W] that if $T_A$ is a tilting module and $B = \text{End}T_A$, then $\text{mod}\hat{A} \cong \text{mod}\tilde{B}$, where $\text{mod}$ denote a stable category in the sense of chapter X in [ARS].

The repetitive algebra $\hat{A}$ was introduced as the Galois covering (see [G]) of the trivial extension $T(A) = A \ltimes DA$ of $A$ by its minimal injective cogenerator $DA$. Let $\nu$ the Nakayama automorphism of $\hat{A}$ and $G = \langle \nu \rangle$. We consider $\hat{A}$ as $k$-category, then we have the Galois cover functor: $F : \hat{A} \to (\hat{A}/G)$, where each element of $\hat{A}$ corresponds to an orbit. This functor induces the push-down functor $F_\chi : \text{mod}\hat{A} \to \text{mod}(\hat{A}/G)$ and pull-up functor $F_\gamma : \text{mod}(\hat{A}/G) \to \text{mod}\hat{A}$, and by 2.2 in [HW] we know that $T(A) \cong \hat{A}/G$.

A $k$-algebra $\hat{A}$ is called $(\nu_A)$-exhaustive, when the push-down functor $F_\chi : \text{mod}\hat{A} \to \text{mod}T(A)$ associated to the Galois cover functor $F : \hat{A} \to T(A)$ is dense.

We say that the $k$-algebra $A$ is of locally finite support, if for each indecomposable projective module $P$, the isomorphism class of the indecomposable projective module $P'$ is such that the number of indecomposable module $M$, with $\text{Hom}_A(P, M) \neq 0$ and $\text{Hom}_A(P', M) \neq 0$ is finite.

In particular, in [LDS] it is show that: If $\hat{A}$ is of locally finite support if and only if the $\text{gldim} A$ (global dimension) is strong and finite, that is, the complexes of the derived category has bounded length.

In [LDS] it was proved that if a $k$-algebra $A$ is locally support finite, then $A$ is $\nu_A$-exhaustive. Now, the following theorem given by Assem and Skowroński in [AS2], establishes a classification of the repetitive algebra $\hat{A}$ which are locally support finite.

**Theorem 2.1.** Let $A$ be a $k$-algebra. The following conditions are equivalent:

i) $\hat{A}$ is tame and exhaustive.

ii) $\hat{A}$ is tame and locally support finite.

iii) There exist an algebra $B$ which is either tilted of Dynkin type,
or representation-infinite tilted of Euclidean type, or tubular, such that \( \hat{A} \cong \hat{B} \).

iv) There exist an algebra \( C \) which is either hereditary of Dynkin or Euclidean type, or tubular canonical, such that \( A \) and \( C \) are tilting-cotilting equivalent.

v) \( \text{mod}\hat{A} \) is cycle-finite.

vi) There exist an algebra \( C \) which is either hereditary of Dynkin or Euclidean type, or tubular canonic such that \( \text{mod}\hat{A} \cong \text{mod}\hat{C} \).

3. Representation of \( T(A) \).

Let \( Q \) be a quiver, \( Sp \) be a subset of the loops of \( Q \), and \( R \) be a set of relations for \( Q \). We call the element of \( Sp \) special loops, the remaining arrows are called ordinary. Let \( R^{Sp} := \{x^2 - x : x \in Sp\} \), and write \( (R) \) for the ideal in \( kQ/(R^{Sp}) \) generated by the element of \( R \) and denote \( J \) the ideal of \( kQ/(R^{Sp}) \) generated by the ordinary arrows.

A triple \( (Q, Sp, R) \) as above is called clannish (see 2.5 in \( [C-B] \)) if the following conditions hold:

1) \( (R) \subset J^2 \)

2) for any vertex of \( Q \) at most 2 arrows start, and at most 2 arrows stop;

3) for every ordinary arrow \( \beta \) there is at most one arrow \( \alpha \) with \( \alpha \beta \notin R \), and at most one arrow \( \gamma \) with \( \beta \gamma \notin R \).

We consider now the following lemma.

**Lemma 3.1.** Let \( A \) be a 2-tubular \( k \)-algebra. Then the trivial extension \( T(A) \) is tame, and the category \( \text{mod}T(A) \) is equivalent to a category \( \text{mod}C \), where \( C \) is clannish.

Proof. Let \( A \) be a 2-tubular \( k \)-algebra. By lemma 2.1, we have that \( D^b(A) \cong D^b(D) \), where \( D \) is given by the quiver in figure 1.

Hence, the ordinary quiver of the trivial extension \( T(D) \) is given by:
with commutativity relations given by: $u \beta = x_1 x$, $v \alpha = x'_1 y$, $xx_{n-1} = \alpha x'_{n-1}$, $x_2 u = x'_{2} x'_1$, $\beta x_{n-1} = y x'_{n-1}$, $x'_2 v = x_2 x_1$ We considered now, the following clannish algebra $C$, given by the following quiver:
where \( q_i(\varepsilon_i) = (\varepsilon_i - K^i_1)(\varepsilon_i - K^i_2) = 0 \) with \( K^i_1 \neq K^i_2 \in k^* \), where \( i = 0, 1, 2 \) and as zero relation we have: \( p_1 p_n = p_3 p_2 = p_2 p_1 = 0 \).

Now, each module \( M \in \text{mod}\mathcal{C} \), has the form: \( M(n_0) = M_{K^0_1} \oplus M_{K^0_2} \) where each \( M_{K^0} \) is a \( k \)-vector space associated to the eigenvalue \( K^0_i \) see 2.6 in [C-B].

Then we can defined a functor \( F: \text{mod}\mathcal{T}(D) \rightarrow \text{mod}\mathcal{C} \) in the following form: Let \( X \in \text{mod}\mathcal{T}(D) \)

\[
F(X)(i_0) = \begin{cases} X_n \oplus X'_n & \text{if } i = n \\ X_2 \oplus X'_2 & \text{if } i = 2 \\ X_1 \oplus X'_1 & \text{if } i = 1 \\ X_i & \text{if } i \neq \{1, 2, n\} \end{cases}
\]

\[
F(X)(p_i) = \begin{cases} (X(x) & X(\beta) \\ -X(\alpha) & -X(y) \\ X(x_1) & X(v) \\ -X(u) & -X(x'_1) \\ X(x_2) & \\ -X(x'_2) & \text{if } i = 3 \\ X(x_{n-1}) & X(x'_{n-1}) & \text{if } i = n \\ X(x_i) & \text{if } i \neq \{1, 2, 3, n\} \end{cases}
\]

\[
F(X)(\varepsilon_i) = \begin{cases} (K^1_1 x_1 & 0 \\ 0 & K^1_2 x'_1) & \text{if } i = 1 \\ (K^2_1 x_2 & 0 \\ 0 & K^2_2 x'_2) & \text{if } i = 2 \\ (K^0_1 x_n & 0 \\ 0 & K^0_2 x'_n) & \text{if } i = n \end{cases}
\]

We define now the functor \( G: \text{mod}\mathcal{C} \rightarrow \text{mod}\mathcal{T}(D) \). Let \( M \in \text{mod}\mathcal{C} \), have that:

\[
p_i = \begin{pmatrix} M^n_{i1} & M^n_{i2} \\ M^n_{21} & M^n_{22} \end{pmatrix}
\]

where the \( M^n_{ij} : M_{K^0_i} = M_t \rightarrow M_{K^0_j} = M_j \) \( i, t, j = 1, 2 \), hence

\[
p_n = \begin{pmatrix} M^n_{1n-1} & M^n_{2n-1} \\ M^n_{1n-1} & M^n_{2n-1} \end{pmatrix}, p_{n-1} = \begin{pmatrix} M^n_{n-1n-2} \\ M^n_{n-1n-2} \end{pmatrix} \text{ and } p_3 = \begin{pmatrix} M^3_{31} \\ M^3_{32} \end{pmatrix}
\]
We defined the functor $G$ in the following form: Let $M \in \text{mod} \mathcal{C}$

$$G(M)(x) = \begin{cases} M_2 & \text{if } x = x_i, \ i = 1, 2, n \\ M_1 & \text{if } x = x'_i, \ i = 1, 2, n \\ M_j & \text{if } x = x_j, \ j \neq \{1, 2, n\} \end{cases}$$

$$G(M)(x) = M_2^{22}(p_1) \quad G(M)(x_1) = M_2^{22}(p_2)$$
$$G(M)(\alpha) = M_1^{11}(p_1) \quad G(M)(u) = M_1^{21}(p_2)$$
$$G(M)(\beta) = -M_2^{11}(p_1) \quad G(M)(v) = -M_2^{21}(p_2)$$
$$G(M)(y) = -M_1^{11}(p_1) \quad G(M)(x'_1) = -M_1^{21}(p_2)$$
$$G(M)(x_2) = M_3^{32}(p_3) \quad G(M)(x_{n-1}) = -M_3^{23}(p_n)$$
$$G(M)(x'_2) = M_3^{31}(p_3) \quad G(M)(x'_{n-1}) = -M_3^{13}(p_3)$$
$$G(M)(x_j) = M_{j+1}(p_j) \quad \text{if } j \neq \{1, 2, 3, n\}$$

Hence, from the relations of Clannish algebra $\mathcal{C}$, it is easy verify that this definition of the functor $G$, defined a module in $\text{mod}T(D)$. By the construction these functors $F$ and $G$ we have that $F \circ G = 1_{\text{mod}T(D)}$ and $G \circ F = 1_{\text{mod} \mathcal{C}}$. But, is known that an algebra Clannish is tame (see [C-B]), then we have that the trivial extension $T(D)$ is tame. Since, a derived equivalence induces a stable equivalence between the trivial extensions (see [HW]), then $\text{mod}T(A) \cong \text{mod}T(D)$ then by Krause (see [Kr]) we have that the trivial extension $T(A)$ is tame.

**Proposition 3.1 (prop. 2 in [DS]).** Let $R$ be a locally bounded $k$-category, and $G$ be the group of the $k$-linear automorphism of $R$ acting freely on the objects of $R$. If $R/G$ is tame, then $R$ also is tame.

**Theorem 3.1.** Let $A$ be a tame concealed algebra, $M$ be an indecomposable module in $\text{mod}A$, and assume $B = A[M]$ is of tame representation. Then the trivial extension $T(B)$ is tame if and only if the module $M$ is regular.

Proof. Assume $T(B)$ is tame. How, $T(B) \cong \hat{B}/(\nu)$, then by Proposition 3.1, we have that $\hat{B} = A[M]$ is tame. Moreover, by lemma 3 in Ringel [R2] if $A[M]$ is tame, then the module $M$ is regular or preinjective.
If the module $M$ is preinjective, it is known that $\widehat{A[M]} \cong \widehat{M}A$, and $[M]A$ is wild by lemma 3 in [R2], hence a full wild subcategory of $[M]A$. Thus, $\widehat{B}$ is wild, which is a contradiction. Therefore, the module $M$ is regular.

Assume $M$ is an indecomposable regular module. We consider two situations:

a) The algebra $A$ is not concealed of type $\widetilde{A}_n$. Hence, we have that the algebra $B$ is tubular or 2-tubular (see 2.2 in [P]).

If $B$ is tubular, then by theorem 2.1 we have that $\widehat{B}$ is tame and exhaustive, therefore the trivial extension $T(B)$ is tame. If $B$ is 2-tubular, then by lemma 3.1 we have that the trivial extension $T(B)$ is tame.

b) $A = \widetilde{A}_n$. We use the table given by Ringel (see th. 3 in [R2]), we have two situations:

b1) The module $M$ is homogeneous. In this situation we have:

1) $(\widetilde{A}_{22}, 1)$; 2) $(\widetilde{A}_{23}, 1)$; 3) $(\widetilde{A}_{24}, 1)$; 4) $(\widetilde{A}_{25}, 1)$; 5) $(\widetilde{A}_{26}, 1)$; 6) $(\widetilde{A}_{2q}, 1)$ if $q \geq 7$

7) $(\widetilde{A}_{33}, 1)$; 8) $(\widetilde{A}_{34}, 1)$; 9) $(\widetilde{A}_{35}, 1)$; 10) $(\widetilde{A}_{36}, 1)$;

11) $(\widetilde{A}_{44}, 1)$

In the situation 1) to 6), these algebras are domestic tubular of type $(2,2,q)$ and that corresponds to the tubular type of $D_{q+2}$. In the situation 7 at 9), this algebras are domestic tubular of type $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$ which corresponds to $E_6$, $E_7$ and $E_8$ respectively. Now, the situation 10) and 11) is also tubular of type $(2,3,6)$ and $(2,4,4)$ corresponding to euclidean type $\widetilde{E}_8$ and $\widetilde{E}_7$ respectively. By theorem 2.1 we have that the trivial extension $T(B)$ is tame.

b2) The module $M$ is not homogeneous. Hence, we have two cases:

b2.1) $(\widetilde{A}_{pq}, p)$, where the module $M$ lies in the mouth of tube of rank $p$. Hence, $(\widetilde{A}_{p+1q})$ which is tubular, therefore the trivial extension $T(B)$ is tame.

b2.2) $(\widetilde{A}_{pq}, 2p)$, where the module $M$ lies in a tube of rank $p$, and regular of length 2. It is easy to see that the proof of theorem 3 in Ringel [R2] if $A_{pq}[N]$ with module $N$ has regular length 2 there exist a sequence of reflexions which takes $B$ into $\widetilde{A}_{pq}[N]$, then $D^b(B)$ group $\cong D^b(\widetilde{A}_{pq}[N])$. 
We consider the $C$ algebra, given by the quiver in the following figure:

![Figure 4.](image)

where the dotted lines is zero relation. We have that $C := \tilde{A}_{p-1}q[M]$ where the module $M$ is defined by:

$$
M := \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & \ldots & 0
\end{bmatrix}
$$

that is lies in the mouth the tube of rank $p - 1$, the algebra $C$ is a tilted algebra of type $\tilde{A}_{pq}$ then there exist a functor $\rho : \mathcal{D}^b(C) \rightarrow \mathcal{D}^b(\tilde{A}_{pq})$, that give a triangular equivalence such that $\rho(M)$ lies in the tube of rank $p$, and is a regular of length 2.

We consider now the algebra $D$ defined by the following quiver:

![Figure 5.](image)
Here the module \( M \) is the same as before. Then, by Barot-Lenzing (see [BL]), we have that:

\[
D^b(D) \sim D^b((A_{pq})[\rho(M)]),
\]

is a one-point extension by module \( \rho(M) \) of regular length 2, then

\[
D^b(B) \sim D^b((A_{pq})[\rho(M)]).
\]

Using the similar construction as the lemma 3.1 with clannish algebras is not difficult to see that \( \text{mod} D \) is equivalent to a clannish algebra \( E \) given by the following quiver:

![Figure 6.](image)

Where \( \epsilon \) is special loop. By Geiss and De la Peña (see 4.4 in [GEP]) the trivial extension \( T(E) \) is tame, thus \( T(D) \) is tame. Since

\[
D^b(D) \cong D^b(B),
\]

therefore \( \text{mod} T(D) \cong \text{mod} T(B) \), then by [Kr] we have that \( T(B) \) is tame.

Before to state the next result, we consider \( A \cong kQ/I \) be an algebra, where \( Q \) is a quiver without oriented cycles. Let \( \chi_A : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z} \) and \( q_A : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z} \) be the quadratic forms defined by:

\[
\chi_A(v) = \sum_{s=0}^{\infty} \sum_{i,j \in Q_0} (-1)^s \text{Ext}_A^s(S_i, S_j)v_i v_j
\]

\[
q_A(v) = \sum_{i \in Q} v_i^2 - \sum_{(i \rightarrow j) \in Q_1} v_i v_j + \sum_{i,j \in Q_0} r(i, j)v_i v_j
\]

where \( v = (v_1, ..., v_n) \), \( r(i, j) = \text{dim} k e_j (I/(IJ + JI)) e_i \), and \( J \) is the ideal generated by arrows of the quiver \( Q \). The quadratic form \( \chi_A \) is called the Euler form of the algebra \( A \), and \( q_A \) is called the Tits form. Its know that if \( \text{gldim} A \leq 2 \), then \( \chi_A = q_A \).
As the consequence of our theorem 3.1, we have the following corollary.

**Corollary 3.1.** Let $A \cong kQ/I$, where $Q$ is a tree, such that Euler form $\chi_A$ is non-negative and $\text{corank} \chi_A \leq 2$. Then the trivial extension $T(A)$ is tame.

Proof. By Barot-De la Peña (see [BP]) we have that the algebra $A$ is domestic tubular, tubular or 2-tubular. Therefore, by the above theorem we have that the trivial extension $T(A)$ is tame.

**References**


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