RIGID SPHERICAL HYPERSONFLACES IN $\mathbb{C}^2$

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Abstract

In this paper we describe explicitly one class of real-analytic hypersurfaces in $\mathbb{C}^2$ rigid and spherical at the origin.
1. Introduction

A real-analytic hypersurface $M$ in $\mathbb{C}^2$ is called rigid if it is given by an equation of the form $r(w, \overline{w}, z, \overline{z}) =: \text{Im}w + F(z, \overline{z}) = 0$,

where $F$ is a real-analytic function such that: $F(0, 0) = \frac{\partial F}{\partial z}(0, 0) = 0$.

In this paper we study the real-analytic hypersurfaces $M$ in $\mathbb{C}^2$ rigid and spherical at the origin, i.e. there exists a local biholomorphic which maps $M$ to the euclidean unit sphere. We note that recently A. Isaev [4] has given a characterization of spherical rigid real hypersurfaces in $\mathbb{C}^n$ ($n \geq 2$) in terms of a certain system of differential equations for a defining function of such hypersurfaces, but this does not permit to describe the spherical rigid real hypersurfaces even if in $\mathbb{C}^2$. Nowadays these hypersurfaces are not known. The only examples have been given by N. Stanton [5] (see also [6]). More recently, B. Coupet and A. Sukhov [3] have described the spherical hypersurfaces of the form: $\text{Im}w + P(z, \overline{z}) = 0$, where $P$ is a non-identically zero subharmonic homogeneous polynomial without purely harmonic terms.

The goal of this paper is to give one description of one class of real-analytic hypersurfaces in $\mathbb{C}^2$ rigid and spherical at the origin.

2. Prelimiaries and results

Let $M$ be a hypersurface in $\mathbb{C}^2$, strictly pseudoconvex at the origin, defined by:

$M =: \{ Rew + \varphi(z, \overline{z}) = 0 \}$, where $\varphi$ is a real-analytic function.

Without any loss of generality, we may assume that $\frac{\partial^2 \varphi}{\partial z \partial \overline{z}}(0, 0) = 1$.

According to a theorem of N. Stanton [5] (see theorem 1.7) there exists an holomorphic change of coordinates $\psi$ of the form $(w, g(z))$ defined in a neighborhood $V$ of the origin and such that $\psi(M \cap V)$ is defined by:

$Rew + |z|^2 + |z|^4 b(z, \overline{z}) = 0$,

where $b$ is a real-analytic function.
Theorem. Let $M$ be a hypersurface in $\mathbb{C}^2$ defined by

$$M = \{ \text{Rew} + \varphi(z, \overline{z}) = 0 \} ,$$

where $\varphi(z, \overline{z}) = |z|^2 + |z|^4 b(z, \overline{z})$ and $b$ being a real-analytic function in a neighborhood of the origin.

Suppose that $\frac{\partial b}{\partial z}(0, 0) = 0$. Then $M$ is spherical at the origin if and only if $\varphi$ is given by one of the functions:

1) $|z|^2$,  
2) $\frac{1}{c} \sin^{-1} \left( c |z|^2 \right)$,  
3) $\frac{1}{c} \sinh^{-1} \left( c |z|^2 \right)$

for some $c \in \mathbb{R}^*$.

Proof. Let $F = (F_1, F_2)$ be a local biholomorphism at the origin which maps $M$ to the euclidean unit sphere:

$$\{ (w, z) \in \mathbb{C}^2 : \rho(w, z) = \text{Rew} + |z|^2 = 0 \} .$$

We may assume that $F_1(0, 0) = F_2(0, 0) = 0$ and $\frac{\partial F_1}{\partial w}(0, 0) = \frac{\partial F_2}{\partial z}(0, 0) = 1$.

By conjugating $F$ with some automorphism of the euclidean unit ball of $\mathbb{C}^2$, we may also assume that $\frac{\partial F_2}{\partial w}(0, 0) = 0$.

The principal idea of the proof is to determine explicitly $F$ by solving a system of partial differential equations. To this end we consider the direct image of the translation vector field $i \frac{\partial}{\partial w} : F_*(i \frac{\partial}{\partial w})$

which is holomorphic tangent vector of the euclidean unit sphere, i.e. $F_*(i \frac{\partial}{\partial w}) = A(w, z) \frac{\partial}{\partial w} + B(w, z) \frac{\partial}{\partial z}$, where $A$ and $B$ are two holomorphic functions in a neighborhood of the origin and such that $\text{Re} \left[ A(w, z) \frac{\partial \rho}{\partial w} + B(w, z) \frac{\partial \rho}{\partial z} \right]$ is identically null on the unit sphere.

We note that the real dimension of the lie algebra of holomorphic tangent vector fields on the unit sphere is equal to 8 (see E. Cartan [1]).

We now proceed in three steps.

First step: $\frac{\partial^n F_1}{\partial z^n}(0, 0) = 0$, \( \forall n \geq 0 \) and $\frac{\partial^n F_2}{\partial z^n}(0, 0) = 0$, \( \forall n \geq 2 \).
We write \( w = u + iv \) and \( \rho(w, z) =: Rev + |z|^2 \).

Let \( A(v, z) =: (−\varphi(z) + iv, z) \) be a parametrization of \( M \). The vector field defined by:

\[
L =: ∂ϕ\overline{w} - \frac{1}{2} ∂z\overline{w}
\]

is tangent to \( M \), so \( L(\rho o F) ≡ 0 \) on \( M \) in a neighbourhood of the origin, which implies the following identity:

\[
(1) \quad \frac{1}{2} \left[ 1\frac{∂F_1}{∂w}oA + (F_2oA)\frac{∂F_2}{∂z}oA \right] \equiv 0
\]

near \( v = 0 \) and \( z = 0 \).

Differentiating (1) with respect to \( z \) to arbitrary order, we get:

\[
(2) \quad \frac{∂}{∂z^n}(0, 0) = 0, \text{ } \forall n ≥ 0 \text{ (See [2] page 47 – 49)}.
\]

We write \( F_2(w, z) \) in the following form:

\[
F_2(w, z) = z + K(z) + \sum_{n ≥ 2} b_n w^n + \sum_{n, m ≥ 1} B_{nm} z^n w^m.
\]

Setting \( v = 0 \) and identifying the pure \( z \) terms in (1), and taking (2) into account we obtain \( K(z) ≡ 0 \).

**Second step:** \( F \) is one of the four following forms:

i) \( F(w, z) = \frac{w}{1 + iΓw} \frac{ze^{-\frac{β_0}{2}w}}{1 + iΓw} \)

ii) \( F(w, z) = \frac{1}{γ} (−tgγw, z\frac{e^{-\frac{β_0}{2}w}}{cosγw}) \)

iii) \( F(w, z) = \frac{i}{k_0} (e^{-ik_0w} - 1), z\frac{e^{-\frac{β_0}{2}w}}{e^{-\frac{β_0}{2}w}} e^{-i\frac{kw}{2}w} \)

iv) \( F(w, z) = \frac{a - 1}{k} e^{kw} - 1, \frac{ae^{kw} - 1}{ae^{kw} - 1}, (a - 1)ze^{-\frac{β_0}{2}w} \frac{e^{kw}}{ae^{kw} - 1} \)

where \( β_0 = b(0, 0), \text{ } Γ \in \mathbb{R}, \text{ } γ \in \mathbb{C}^*, \text{ } γ^2 \in \mathbb{R}^*, \text{ } k_0 \in \mathbb{R}^*, \text{ } k \in \{R^*, iR^*\}, a \in \mathbb{C}^* \text{ and } a ≠ 1, \text{ } |a|^2 = 1 \text{ if } k \in \mathbb{R}^* \text{ and } a \in \mathbb{R}^* \text{ if } k \in i\mathbb{R}^* \).

First, we shall prove that \( F_1(w, z) = F_1(w) \), i.e. \( F_1 \) depends only on \( w \).
We consider the holomorphic vector field $F_*(i \frac{\partial}{\partial w})$ which is defined in a neighbourhood of the origin. Since $F_*(i \frac{\partial}{\partial w})$ is tangent to the euclidean unit sphere: \( \{(w, z) \in \mathbb{C}^2 : Rew + |z|^2 = 0\} \), it may be written as a real linear combination of the following fields:

\[
\begin{align*}
X_1 &= i \frac{\partial}{\partial w} \\
X_2 &= -2z \frac{\partial}{\partial w} + \frac{\partial}{\partial z} \\
X_3 &= 2iz \frac{\partial}{\partial w} + i \frac{\partial}{\partial\bar{z}} \\
X_4 &= 2w \frac{\partial}{\partial w} + z \frac{\partial}{\partial z} \\
X_5 &= i z \frac{\partial}{\partial\bar{z}} \\
X_6 &= 2izw \frac{\partial}{\partial w} + (2iz^2 - iw) \frac{\partial}{\partial z} \\
X_7 &= 2zw \frac{\partial}{\partial w} + (2z^2 + w) \frac{\partial}{\partial z} \\
X_8 &= -iw^2 \frac{\partial}{\partial w} - izw \frac{\partial}{\partial z}
\end{align*}
\]

Then there are real numbers $\alpha_1, ..., \alpha_8$ such that:

\[
F_*(i \frac{\partial}{\partial w}) = \sum_{j=1}^{8} \alpha_j X_j
\]

We note $\left[F_*(i \frac{\partial}{\partial w})\right]_{(w, z)} =: A(w, z) \frac{\partial}{\partial w} + B(w, z) \frac{\partial}{\partial z}$.

Then $A(w, z) = i\alpha_1 + 2(-\alpha_2 + i\alpha_3)z + 2\alpha_4 w - i\alpha_8 w^2 + 2\lambda w z$ and $B(w, z) = (\alpha_2 + i\alpha_3) + \mu z + 2\lambda z^2 + \lambda w - i\alpha_8 wz$

where $\mu = \alpha_4 + i\alpha_5$ and $\lambda = \alpha_7 + i\alpha_6$.

On the other hand we have:

\[
\left[F_*(i \frac{\partial}{\partial w})\right]_{F(w, z)} = i \frac{\partial F_1}{\partial w}(w, z) \frac{\partial}{\partial w} + i \frac{\partial F_2}{\partial w}(w, z) \frac{\partial}{\partial z}.
\]

Then, we obtain:

\[(3) \quad i \frac{\partial F_1}{\partial w}(w, z) = (\alpha_0 F)(w, z)\]
and

\[ i \frac{\partial F_2}{\partial w}(w, z) = (BoF)(w, z) \]

Since \( \frac{\partial F_1}{\partial w}(0, 0) = 1 \) and \( \frac{\partial F_2}{\partial w}(0, 0) = 0 \), then the identities (3) and (4) become:

\[ i \frac{\partial F_1}{\partial w} = i + 2\alpha_4 F_1 - i\alpha_8 F_1^2 + 2\lambda F_1 F_2 \]

and

\[ i \frac{\partial F_2}{\partial w} = \mu F_2 + 2\lambda F_2^2 + \overline{\lambda} F_1 - i\alpha_8 F_1 F_2 \]

where \( \mu = \alpha_4 + i\alpha_5 \) and \( \lambda = \alpha_7 + i\alpha_6 \).

We momentarily admit that \( \lambda = \frac{1}{2} \frac{\partial b}{\partial z}(0, 0) \) and \( \alpha_5 = -\frac{\beta_0}{2} \), the proof will be given in the end of this paper.

By hypothesis \( \frac{\partial b}{\partial z}(0, 0) = 0 \), then \( \lambda = 0 \), so the identities (5) and (6) become:

\[ \frac{\partial F_1}{\partial w} = 1 - 2i\alpha_4 F_1 - \alpha_8 F_1^2 \]

and

\[ \frac{\partial F_2}{\partial w} = -i\mu F_2 - \alpha_8 F_1 F_2 \]

where \( \mu = \alpha_4 - i\frac{\beta_0}{2} \).

Since \( F_2(0, 0) = \frac{\partial F_2}{\partial w}(0, 0) = 0 \), from (8) we deduce by induction that

\[ \frac{\partial^n F_2}{\partial w^n}(0, 0) = 0, \forall n \geq 0, \text{ i.e. } F_2(w, 0) \equiv 0, \text{ this implies that:} \]

\[ F_1(w, z) = F_1(w) \] (See[2] page 47).

We are now in order to solve the system of differential equations (7) and (8).

Let us recall: (9) \( F(0, 0) = (0, 0) \), \( \frac{\partial F_1}{\partial w}(0, 0) = \frac{\partial F_2}{\partial z}(0, 0) = 1 \) and

\[ \frac{\partial^n F_2}{\partial z^n}(0, 0) = 0, \forall n \geq 2. \]

There are two cases to consider.

**Case 1. \( \alpha_4 = 0 \)**

We suppose that \( \alpha_8 = 0 \). In this case (7) and (8) become:
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(10) \[ \frac{\partial F_1}{\partial w} = 1 \]
(11) \[ \frac{\partial F_2}{\partial w} = -\frac{\beta_0}{2} F_2 \]

then $F_1(w, z) = w$ and $F_2(w, z) = h(z)e^{-\frac{\beta_0}{2}w}$ where $h$ is a holomorphic function. Taking (9) into account we obtain $F_2(w, z) = ze^{-\frac{\beta_0}{2}w}$, this corresponds to the case i).

We suppose now that $\alpha_8 \neq 0$. Let $\eta \in \mathbb{C}^*$ such that: $\eta^2 = \frac{1}{\alpha_8}$ a particular solution of the Riccati equation (7). Then it is easy to show that:

(12) \[ F_1(w, z) = \frac{1}{\gamma} t g \gamma w, \text{ where } \gamma = \frac{1}{i \eta}; \gamma^2 = -\alpha_8. \]

Replacing $F_1(w, z)$ by its expression (12) into (8) and observing that $\alpha_8 = -\gamma$, then we obtain:

(13) \[ \frac{\partial F_2}{\partial w} = (-\frac{\beta_0}{2} + \gamma t g \gamma w)F_2. \]

From (13) and (9) we obtain:

$F_2(w, z) = ze^{-\frac{\beta_0}{2}w} \frac{1}{\cos \gamma w}$, this corresponds to the case ii).

**Case 2. $\alpha_4 \neq 0$**

We proceed analogously to the first case.

First we suppose that $\alpha_8 = 0$. From (7), (8) and (9) we obtain:

$F_1(w, z) = \frac{i}{k_0} (e^{-ik_0 w} - 1)$

and $F_2(w, z) = ze^{-\frac{\beta_0}{2}w}e^{-i\frac{k_0}{2}w}$

where $k_0 = 2\alpha_4$, this corresponds to the case iii).

Now, we suppose that $\alpha_8 \neq 0$. Let $\eta \in \mathbb{C}^*$ such that: $\alpha_8 \eta^2 + 2i\alpha_4 \eta = 1$, a particular solution of the Riccati equation (7).

We put: $k = 2(\alpha_8 \eta + i\alpha_4)$.

If $k = 0$, in this case, we obtain from (7), (8) and (9):

$F_1(w, z) = \frac{w}{1 + i \Gamma w}$ and $F_2(w, z) = \frac{ze^{-\frac{\beta_0}{2}w}}{1 + i \Gamma w}$

where $\Gamma = \alpha_4$, this corresponds to the case i).
If \( k \neq 0 \), then from (7) we deduce:

\[
F_1(w, z) = \frac{1}{\delta e^{kw} - \frac{\alpha_8}{k}} + \eta; \quad \delta \in \mathbb{C}^*
\]

or also

\[
F_1(w, z) = \frac{a - 1}{k} \frac{e^{kw} - 1}{ae^{kw} - 1}, \quad \text{where} \quad a = \frac{\delta k}{\alpha_8}.
\]

Replacing \( F_1(w, z) \) by its expression (14) into (8) and observing that \( \alpha_8 \eta + i\alpha_4 = \frac{k}{2} \), then, we obtain:

\[
\frac{\partial F_2}{\partial w} = \left( -\frac{\beta_0}{2} - \frac{k}{2} \frac{ae^{kw} + 1}{ae^{kw} - 1} \right) F_2
\]

From (16) and (9) we deduce:

\[
F_2(w, z) = (a - 1)ze^{-\frac{\beta_0}{2}} \frac{e^{\frac{k}{2}w}}{ae^{kw} - 1}
\]

Now, we shall prove that \( k \in \{ \mathbb{R}^*, i\mathbb{R}^* \} \), \( a \in \mathbb{C}^* \) and \( a \neq 1 \), \( |a|^2 = 1 \) if \( k \in \mathbb{R}^* \) and \( a \in \mathbb{R}^* \) if \( k \in i\mathbb{R}^* \).

\( F \) is a local biholomorphism, so, \( a \neq 1 \). Since the image \( F(M) \) is contained in the unit sphere:

\[
\{(w, z) \in \mathbb{C}^2 : Re w + |z|^2 = 0\} \quad \text{near} \quad (0, 0), \quad \text{hence} \quad Re F_1(w) = 0 \quad \text{for} \quad Re w = 0, \quad \text{then we obtain}:
\]

\[
Re \left( a - 1 \right) + \left( Re(\bar{a}) \left( \frac{a - 1}{k} \right) \right) e^{2vRe i k} - \left( \frac{a - 1}{k} + a \left( \frac{a - 1}{k} \right) \right) e^{ik} - \left( \bar{a} \left( \frac{a - 1}{k} \right) + \left( \frac{a - 1}{k} \right) \right) e^{-i\bar{k}} \equiv 0 \quad \text{for} \quad v \quad \text{near} \quad 0.
\]

First we prove that \( k \in \{ \mathbb{R}^*, i\mathbb{R}^* \} \). Assume, to the contrary, that \( k \in \mathbb{C} \setminus \{ \mathbb{R}^*, i\mathbb{R}^* \} \), then \( Re(ik) \neq 0 \) and \( ik \neq \bar{k} \). From (18) we obtain:

\[
Re \left( a - 1 \right) = 0 \quad \text{and} \quad \frac{a - 1}{k} + a \left( \frac{a - 1}{k} \right) = 0.
\]
Thus, it follows that: 

\[ \frac{1}{k} |a - 1|^2 = 0, \text{ then } a = 1, \text{ this is a contradiction, so, } k \in \{ \mathbb{R}^*, i \mathbb{R}^* \}. \]

From (18) we deduce: \(|a|^2 = 1 \text{ if } k \in \mathbb{R}^* \text{ and } a \in \mathbb{R}^* \text{ if } k \in i \mathbb{R}^* \).

**Third step: conclusion**

We return to the second step. Let us first prove that \( \beta_0 = 0 \).

According to the result of N. Stanton [5] (theorem 1.7), it suffices to prove that \( \varphi(z, \overline{z}) = \varphi(|z|^2) \), \( (\beta_0 \text{ is the coefficient of } |z|^4) \).

There are four cases to consider. For example, we suppose that \( F \) is given by ii), i.e. \( F(w, z) = \left( \frac{1}{\gamma} tg \gamma w, \frac{z \theta}{\cos \gamma w} \right) \), where \( \gamma \in \mathbb{C}^\ast \) and \( \gamma^2 \in \mathbb{R}^\ast \).

So, \( \gamma^2 \in \mathbb{R}^\ast \text{ then } \gamma \in \mathbb{R}^\ast \text{ or } \gamma = i\gamma_0, \gamma_0 \in \mathbb{R}^\ast \).

Since the image \( F(M) \) is contained in the unit sphere:

\[ \{(w, z) \in \mathbb{C}^2 : \text{Re} w + |z|^2 = 0\} \text{ near } (0, 0), \text{ we have} \]

\[ h(\varphi(z, \overline{z})) = |z|^2, \text{ where } h(x) = \begin{cases} 
\frac{1}{2\gamma}(\sin 2\gamma x)e^{-\beta_0 x} & \text{if } \gamma \in \mathbb{R}^\ast \\
\frac{1}{2\gamma_0}(sh 2\gamma_0 x)e^{-\beta_0 x} & \text{if } \gamma = i\gamma_0
\end{cases} \]

Hence \( \varphi(z, \overline{z}) = h^{-1}(|z|^2) \text{ near } 0. \)

So, \( \beta_0 = 0 \) and

\[ \varphi(z, \overline{z}) = \begin{cases} 
\frac{1}{2\gamma} \sin^{-1} \left( 2\gamma |z|^2 \right) & \text{if } \gamma \in \mathbb{R}^\ast \\
\frac{1}{2\gamma_0} sh^{-1} \left( 2\gamma_0 |z|^2 \right) & \text{if } \gamma = i\gamma_0
\end{cases} \]

By following the same way we obtain the other cases.

To end the proof of the theorem it remains to show that:

\[ \lambda = \frac{1}{2i} \frac{\partial b}{\partial z} (0, 0) \quad \text{and} \quad \alpha_5 = -\frac{\beta_0}{2}. \]

Let’s return to the identities (5) and (6). From the first step we have:
\[
\frac{\partial^n F_1}{\partial z^n}(0,0) = 0, \quad \forall n \geq 0, \text{ then, from (5) we deduce that:}
\]
\[
\frac{\partial^{n+1} F_1}{\partial z^{n+1} \partial w}(0,0) = 0, \quad \forall n \geq 1, \text{ So, in a neighbourhood fo the origin, we can write:}
\]
\[
(19) \quad F_1(w,z) = w + \sum_{n \geq 2} a_n w^n + w^2 \sum_{n \geq 1} A_{n} z^n + \sum_{q \geq 3} w^q \sum_{n \geq 1} A_{nq} z^n
\]
\[
(20) \quad F_2(w,z) = z + \sum_{n \geq 2} b_n w^n + w \sum_{n \geq 1} B_{n1} z^n + w^2 \sum_{n \geq 1} B_{n2} z^n + \sum_{q \geq 3} w^q \sum_{n \geq 1} B_{nq} z^n
\]

The idea to prove \( \lambda = \frac{1}{2i} \frac{\partial b}{\partial z}(0,0) \) and \( \alpha_5 = -\frac{\beta_0}{2} \) is to observe that the terms of degree less or equal to 4 on \( z, \bar{z} \) in the left hand-side of (1) are null.

First, we observe that from (5) and (19) we have:

\[
(21) \quad A_{12} = -i\lambda \text{ and } a_2 = -i\alpha_4
\]

Next, from (6) and (20) we have:

\[
(22) \quad b_2 = -i\frac{\lambda}{2}, \quad B_{21} = -2i\lambda \text{ and } B_{11} = \alpha_5 - i\alpha_4
\]

We are now in order to collect the terms of degree less or equal to 4 on \( z, \bar{z} \) in the left hand-side of (1).

Let us write \( b(z,\bar{z}) = \beta_0 + \beta_1 z + \bar{\beta}_1 \bar{z} + ... \)

The terms of degree less or equal to 4 in \( \frac{\partial \varphi}{\partial z} \) are:

\[
\bar{z} + 2\beta_0 |z|^2 + 2\bar{\beta}_1 |z|^2 |z|^2 + 3\beta_1 |z|^4.
\]

Since \( \frac{\partial \varphi}{\partial z}(0,0) = 0 \), it suffices to collect the terms of degree less or equal to 3 on \( z, \bar{z} \) in \( \left[ \frac{1}{2} \frac{\partial F_1}{\partial w} \circ A + \left( \frac{\partial F_2}{\partial w} \circ A \right) \right] \), which are:

\[
\frac{1}{2} + (B_{11} - a_2) |z|^2 + (B_{21} - A_{12}) \bar{z} |z|^2 - 2b_2 \bar{z} |z|^2.
\]

the terms of degree less or equal to 4 on \( z, \bar{z} \) in

\[
\frac{\partial \varphi}{\partial z} \left[ \frac{1}{2} \frac{\partial F_1}{\partial w} \circ A + \left( \frac{\partial F_2}{\partial w} \circ A \right) \right]
\]
are:

\[ \frac{1}{2} z + (\beta_0 + B_{11} - a_2) \overline{z} |z|^2 + (\overline{\beta}_1 - 2b_2) z^2 |z|^2 + (B_{21} - A_{12} + \frac{3}{2} \beta_1) |z|^4 \]

On the other hand, the terms of degree less or equal to 4 on \( z, \overline{z} \) in

\[ \frac{1}{2} \left[ \frac{1}{2} \frac{\partial F_1}{\partial z} \frac{oA}{oA} + \left( F_2 \frac{oA}{oA} \right) \frac{\partial F_2}{\partial z} \frac{oA}{oA} \right] \]

are:

\[ \frac{1}{2} z - \overline{z} |z|^2 \text{Re} B_{11} - \frac{1}{2} \overline{B}_{21} \overline{z}^2 |z|^2 + \frac{1}{2} \left( \overline{B}_2 - 2B_{21} + \frac{1}{2} A_{12} \right) |z|^4. \]

Finally, the terms of degree less or equal to 4 on \( z, \overline{z} \) in the left hand-side of (1) are:

\[ (\beta_0 + B_{11} - a_2 + \text{Re} B_{11}) \overline{z} |z|^2 + (\overline{\beta}_1 - 2b_2 + \frac{1}{2} \overline{B}_{21}) \overline{z}^2 |z|^2 \]

\[ + (2B_{21} - \frac{5}{4} A_{12} + \frac{3}{2} \beta_1 - \frac{1}{2} \overline{B}_2) |z|^4. \]

These terms are null, then:

(23) \[ \beta_0 + B_{11} - a_2 + \text{Re} B_{11} = 0 \]

and

(24) \[ \overline{\beta}_1 - 2b_2 + \frac{1}{2} \overline{B}_{21} = 0 \]

From (21), (22) and (23) we obtain: \( \beta_0 = -2 \text{Re} B_{11} = -2a_5 \)

and

From (22) and (24) we obtain: \( \lambda = \frac{\beta_1}{2i} = \frac{1}{2i} \frac{\partial b}{\partial z}(0,0) \).

This ends the proof of the theorem.

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