The Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold

D. G. PRAKASHA¹

H. HARISH^{2,3}

P. VEERESHA⁴

VENKATESHA⁵

¹ Department of Mathematics, Davangere
University, Davangere - 577 007, India.
prakashadg@gmail.com

² Department of Mathematics,
Mahaveera College, Mudbidri, India.

³ Department of Mathematics, Karnatak
University, Dharwad - 580 003, India.
harishjonty@gmail.com

⁴ Department of Mathematics, CHRIST
(Deemed to be University), Bengaluru -
560029, India.
pundikala.veeresha@christuniversity.in;
viru0913@gmail.com

⁵ Department of Mathematics, Kuvempu
University, Shankaraghatta - 577 4511,
Shimoga District, India.
vensmath@gmail.com

ABSTRACT

The object of the paper is to study a type of canonical linear connection, called the Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold.

RESUMEN

El objetivo de este artículo es estudiar un tipo de conexión lineal canónica, llamada la conexión canónica paracontacto de Zamkovoy en una variedad para-Kenmotsu.

Keywords and Phrases: Para-Kenmotsu manifold; Zamkovoy canonical paracontact connection; local ϕ -symmetry; local ϕ -Ricci symmetry; recurrent; η -Einstein manifold.

2020 AMS Mathematics Subject Classification: 53C21, 53C25, 53C44.

1 Introduction

In recent years, many authors started to the study of paracontact geometry due to its unexpected relation with the most activated contact geometry. As a result of this, in 1985, S. Kaneyuki and F. L. Williams [9] introduced the notion of paracontact metric manifold as a natural counter part of the well known contact metric manifold. Since then, several authors studied these manifolds by focusing on various special cases. A systematic study of paracontact metric manifolds and their subclasses were carried out by S. Zamkovoy [25] by emphasizing similarities and differences with respect to the contact case. Further, the notion of para-Kenmotsu manifold was introduced by J. Welyczko [23] for 3-dimensional normal almost paracontact metric structures. This structure is an analogy of Kenmotsu manifold [10] in paracontact geometry. Again the similar notion called P-Kenmotsu manifold was studied by B. B. Sinha and K. L. Sai Prasad [20] and they obtained many results. At this point, we refer the papers [1, 4, 14, 15, 16, 26] and the references therein to reader for a wide and detailed overview of the results on para-Kenmotsu manifolds.

In the context of para-Kenmotsu geometry, author A. M. Blaga [2] studied certain canonical linear connections (Levi-Civita, Schouten-van Kampen, Golab and Zamkovoy canonical paracontact connections) with a special view towards ϕ -conjugation. Some properties of generalized dual connections of the above said canonical linear connections on a para-Kenmotsu manifold was also studied in [3]. As a continuation of this, we are considering one of such canonical linear connection on a para-Kenmotsu manifold. So we undertake the study of Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold. This connection on a paracontact manifolds was adapted and studied rigorously by S. Zamkovoy [25]. This connection plays the role of the (generalized) Tanaka-Webster connection [22] in paracontact geometry. The main feature of this connection is that, it is metrical but not symmetrical. Throughout the paper, we refer the canonical linear connection as *Zamkovoy canonical paracontact connection*.

On the other hand, the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent. In 1977, T. Takahashi [21] introduced the notion of local ϕ -symmetry on a Sasakian manifold as a weaker version of local symmetry of such a manifold. Since then, several authors studied this notion on various structures and their generalizations or extension in [6, 7, 12, 13, 17, 18, 20]. A para-Kenmotsu manifold is said to be locally ϕ -symmetric if its curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X, Y)U) = 0 \quad (1.1)$$

for any vector fields X, Y, U, W orthogonal to ξ on M , where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g .

Recently, U. C. De and A. Sarkar [5] introduced the notion of local ϕ -Ricci symmetry on a Sasakian manifold. Further, this notion was studied by S. Ghosh and U. C. De [8] in the context of (κ, μ) -

contact metric manifolds and obtained interesting results. A para-Kenmotsu manifold M is said to be locally ϕ -Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2((\nabla_W Q)X) = 0,$$

for any vector fields X, W orthogonal to ξ on M and $S(X, W) = g(QX, W)$.

The object of the present paper is to study the Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold. This paper is organized as follows: Section 2 is devoted to preliminaries on para-Kenmotsu manifolds. In section 3, we give a brief account of information regarding the Zamkovoy canonical paracontact connection ∇^Z on a para-Kenmotsu manifold and obtain a relationship between the Levi-Civita connection ∇ and the Zamkovoy canonical paracontact connection ∇^Z . In section 4, we characterize locally ϕ -symmetric and locally concircular ϕ -symmetric para-Kenmotsu manifolds with respect to the connection ∇^Z . It is prove that the notion of local ϕ -symmetry (also, locally concircular ϕ -symmetry) with respect to the connections ∇^Z and ∇ are equivalent. Section 5, covers the study of locally ϕ -Ricci symmetric para-Kenmotsu manifold with respect to the connection ∇^Z and prove that a para-Kenmotsu manifold is locally ϕ -symmetric with respect to the connection ∇^Z , then the manifold is Ricci symmetric and hence it is an Einstein manifold. A para-Kenmotsu manifold whose curvature tensor is covariant constant with respect to the connection ∇^Z and the manifold is recurrent with respect to the connection ∇ is studied in section 6 and shown that in this situation the manifold is η -Einstein manifold. Finally, we construct an example of a 3-dimensional para-Kenmotsu manifold admitting the connection ∇^Z to illustrate some results.

2 Preliminaries

Let M be an n -dimensional differentiable manifold, n is odd, with an almost paracontact structure (ϕ, ξ, η) , that is, ϕ is a $(1, 1)$ -tensor field, ξ is a vector field, and η is a 1-form such that

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (2.1)$$

$$\phi\xi = 0, \quad \eta \cdot \phi = 0, \quad \text{rank}(\phi) = n - 1. \quad (2.2)$$

Let g be a pseudo-Riemannian metric compatible with (ϕ, ξ, η) , that is,

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \quad (2.3)$$

for any vector fields $X, Y \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M , then the manifold is said to be an almost paracontact metric manifold. From (2.3) it can be easily deduce that

$$g(X, \phi Y) = -g(\phi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X), \quad (2.4)$$

for any vector fields $X, Y \in \chi(M)$. An almost paracontact metric manifold becomes a paracontact metric manifold [25] if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric g and is denoted by (M, g) . If moreover,

$$(\nabla_X \phi)Y = g(X, \phi Y)\xi - \eta(Y)\phi X, \quad (2.5)$$

where ∇ denotes the pseudo-Riemannian connection of g holds, then (M, g) is called an para-Kenmotsu manifold. From (2.5), it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \quad (2.6)$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \quad (2.7)$$

Moreover, in a para-Kenmotsu manifold (M, g) of dimension n , the curvature tensor R and the Ricci tensor S satisfy [23]:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.8)$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (2.9)$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad (2.10)$$

$$S(X, \xi) = -(n-1)\eta(X), \quad (2.11)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \quad (2.12)$$

for any vector fields $X, Y, Z \in \chi(M)$.

A para-Kenmotsu manifold M is said to be an η -Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X, W) = ag(X, W) + b\eta(X)\eta(W),$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M reduces to an Einstein manifold with some constant a .

3 Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold

In the following, we consider a connection ∇^Z on an almost paracontact metric manifold using the Levi-Civita connection ∇ of the structure [25]:

$$\nabla_X^Z Y = \nabla_X Y + (\nabla_X \eta)Y \cdot \xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \quad (3.1)$$

If we use (2.6) and (2.7) in (3.1), we obtain

$$\nabla_X^Z Y = \nabla_X Y + g(X, Y)\xi - \eta(Y)X + \eta(X)\phi Y, \quad (3.2)$$

for any vector fields $X, Y \in \chi(M)$. We call the connection ∇^Z defined by (3.2) on a para-Kenmotsu manifold, the *Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold*.

The expression for the curvature tensor R_{∇^Z} with respect to the connection ∇^Z is defined by

$$R_{\nabla^Z}(X, Y)U = \nabla_X^Z \nabla_Y^Z U - \nabla_Y^Z \nabla_X^Z U - \nabla_{[X, Y]}^Z U.$$

Then, in a para-Kenmotsu manifold, we have

$$R_{\nabla^Z}(X, Y)U = R(X, Y)U + g(Y, U)X - g(X, U)Y, \quad (3.3)$$

where $R(X, Y)U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X, Y]} U$, is the curvature tensor of M with respect to the connection ∇ . The expression (3.3) is treated as the curvature tensor of a para-Kenmotsu manifold with respect to the connection ∇^Z .

Proposition 3.1. *A para-Kenmotsu manifold is Ricci-flat with respect to the Zamkovoy canonical paracontact connection if and only if it is an Einstein manifold of the form $S(Y, U) = -(n - 1)g(Y, U)$.*

Proof. In a para-Kenmotsu manifold M , the Ricci tensor S_{∇^Z} and scalar curvature r_{∇^Z} of the Zamkovoy canonical paracontact connection ∇^Z are defined by

$$S_{\nabla^Z}(Y, U) = S(Y, U) + (n - 1)g(Y, U), \quad (3.4)$$

$$r_{\nabla^Z} = r + n(n - 1), \quad (3.5)$$

where S and r denote the Ricci tensor and scalar curvature of Levi-Civita connection ∇ , respectively.

Remark 3.2. *For a para-Kenmotsu manifold M with respect to the Zamkovoy canonical paracontact connection ∇^Z :*

- (a) *The curvature tensor R_{∇^Z} is given by (3.3),*
- (b) *The Ricci tensor S_{∇^Z} is given by (3.4),*
- (c) $R_{\nabla^Z}(X, \xi)U = R_{\nabla^Z}(\xi, Y)U = R_{\nabla^Z}(X, Y)\xi = 0$,
- (d) $R'_{\nabla^Z}(X, Y, U, V) + R'_{\nabla^Z}(X, Y, V, U) = 0$,
- (e) $R'_{\nabla^Z}(X, Y, U, V) + R'_{\nabla^Z}(Y, X, V, U) = 0$,
- (f) $R'_{\nabla^Z}(X, Y, U, V) - R'_{\nabla^Z}(U, V, X, Y) = 0$,
- (g) $R_{\nabla^Z}(X, \xi)U = R_{\nabla^Z}(\xi, Y)U = R_{\nabla^Z}(X, Y)\xi = 0$,
- (h) $S_{\nabla^Z}(Y, \xi) = 0$,
- (i) *The Ricci tensor S_{∇^Z} is symmetric,*
- (j) *The scalar curvature r_{∇^Z} is given by (3.5).*

Next, suppose that a para-Kenmotsu manifold is Ricci flat with respect to the Zamkovoy canonical paracontact connection. Then from (3.4) we get

$$S(Y, U) = -(n - 1)g(Y, U). \quad (3.6)$$

Conversely, if the manifold is an Einstein manifold of the form $S(Y, U) = -(n-1)g(Y, U)$, then from (3.4) it follows that $S_{\nabla}(Y, U) = 0$: $\square$

Proposition 3.3. *If in a para-Kenmotsu manifold the curvature tensor of the Zamkovoy canonical paracontact connection vanishes, then the sectional curvature of the plane determined by two vectors $X, Y \in \xi^\perp$ is -1 .*

Proof. Let $\xi^\perp$ denote the $(n-1)$ -dimensional distribution orthogonal to ξ in a para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection whose curvature tensor vanishes. Then for any $X \in \xi^\perp$, $g(X, \xi) = 0$ or, $\eta(X) = 0$. Now we shall determine the sectional curvature $'R$ of the plane determined by the vectors $X, Y \in \xi^\perp$. Taking inner product on both sides of (3.3) with X and then for $U = Y$, we have

$$R_{\nabla^Z}(X, Y, Y, X) = R(X, Y, Y, X) + g(Y, Y)g(X, X) - g(X, Y)g(X, Y). \quad (3.7)$$

Putting $R_{\nabla^Z} = 0$ in (3.7) we get

$$'R(X, Y) = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} = -1.$$

This proves the required result. $\square$

4 Local ϕ -symmetry and local concircular ϕ -symmetry with respect to the connections ∇^Z and ∇

Definition 4.1. *A para-Kenmotsu manifold is said to be locally ϕ -symmetric with respect to the Zamkovoy canonical paracontact connection ∇^Z if its curvature tensor R_{∇^Z} with respect to the connection ∇^Z satisfies the condition*

$$\phi^2((\nabla_W^Z R_{\nabla^Z})(X, Y)U) = 0, \quad (4.1)$$

for any vector fields X, Y, U, W orthogonal to ξ .

Proposition 4.2. *A para-Kenmotsu manifold is locally ϕ -symmetric with respect to the Zamkovoy canonical paracontact connection ∇^Z if and only if it is so with respect to the Levi-Civita connection ∇ .*

Proof. Let us suppose that a para-Kenmotsu manifold M is locally ϕ -symmetric with respect to the Zamkovoy canonical paracontact connection ∇^Z . Then, by the help of (3.2), (4.1) simplifies as follow

$$\begin{aligned} (\nabla_W^Z R_{\nabla^Z})(X, Y)U &= (\nabla_W R_{\nabla^Z})(X, Y)U + g(W, R_{\nabla^Z}(X, Y)U)\xi \\ &\quad - \eta(R_{\nabla^Z}(X, Y)U)W + \eta(W)\phi R_{\nabla^Z}(X, Y)U. \end{aligned} \quad (4.2)$$

By virtue of $\eta(R_{\nabla^Z}(X, Y)U) = 0$, (4.2) reduces to

$$\begin{aligned} & (\nabla_W^Z R_{\nabla^Z})(X, Y)U \\ &= (\nabla_W R_{\nabla^Z})(X, Y)U + g(W, R_{\nabla^Z}(X, Y)U)\xi + \eta(W)\phi R_{\nabla^Z}(X, Y)U. \end{aligned} \quad (4.3)$$

Now covariant differentiation of (3.3) with respect to W , we obtain

$$(\nabla_W R_{\nabla^Z})(X, Y)U = (\nabla_W R)(X, Y)U. \quad (4.4)$$

Using (4.4) in (4.3), we get

$$\begin{aligned} & (\nabla_W^Z R_{\nabla^Z})(X, Y)U \\ &= (\nabla_W R)(X, Y)U + \{R'(X, Y, U, W) + g(Y, U)g(X, W) - g(X, U)g(Y, W)\}\xi \\ &+ \eta(W)\{\phi R(X, Y)U + g(Y, U)\phi X - g(X, U)\phi Y\}. \end{aligned} \quad (4.5)$$

Applying ϕ^2 on both sides of (4.5); then using (2.1) and (2.2), we obtain

$$\begin{aligned} \phi^2(\nabla_W^Z R_{\nabla^Z})(X, Y)U &= \phi^2(\nabla_W R)(X, Y)U \\ &+ \eta(W)\{\phi R(X, Y)U + g(Y, U)\phi X - g(X, U)\phi Y\}. \end{aligned} \quad (4.6)$$

If we consider X, Y, U, W orthogonal to ξ , (4.6) gives to

$$\phi^2((\nabla_W^Z R_{\nabla^Z})(X, Y)U) = \phi^2((\nabla_W R)(X, Y)U).$$

It completes the proof. $\square$

Definition 4.3. For an n -dimensional ($n > 1$) para-Kenmotsu manifold the concircular curvature tensor C_{∇^Z} with respect to the Zamkovoy canonical paracontact connection is defined by

$$C_{\nabla^Z}(X, Y)U = R_{\nabla^Z}(X, Y)U - \frac{r_{\nabla^Z}}{n(n-1)}[g(Y, U)X - g(X, U)Y]. \quad (4.7)$$

where R_{∇^Z} and r_{∇^Z} are the Riemannian curvature tensor and scalar curvature with respect to the connection ∇^Z , respectively.

Using (3.3) and (3.5) in (4.7), we get

$$C_{\nabla^Z}(X, Y)U = C(X, Y)U, \quad (4.8)$$

where

$$C(X, Y)U = R(X, Y)U - \frac{r}{n(n-1)}[g(Y, U)X - g(X, U)Y] \quad (4.9)$$

is the concircular curvature tensor [24] with respect to the Levi-Civita connection ∇ . Thus, the concircular curvature tensor with respect to the connections ∇^Z and ∇ are equal.

Definition 4.4. A para-Kenmotsu manifold is said to be locally concircular ϕ -symmetric with respect to the Zamkovoy canonical paracontact connection ∇^Z if its concircular curvature tensor C_{∇^Z} with respect to the connection ∇^Z satisfies the condition

$$\phi^2((\nabla_W^Z C_{\nabla^Z})(X, Y)U) = 0, \quad (4.10)$$

for any vector fields X, Y, U, W orthogonal to ξ .

Proposition 4.5. A para-Kenmotsu manifold is locally concircular ϕ -symmetric with respect to the Zamkovoy canonical paracontact connection ∇^Z if and only if it is so with respect to the Levi-Civita connection ∇ .

Proof. If a para-Kenmotsu manifold M is locally concircular ϕ -symmetric with respect to the Zamkovoy canonical paracontact connection ∇^Z , then using (3.2), (4.10) simplifies to

$$\begin{aligned} (\nabla_W^Z C_{\nabla^Z})(X, Y)U &= (\nabla_W C_{\nabla^Z})(X, Y)U + g(W, C_{\nabla^Z}(X, Y)U)\xi \\ &\quad - \eta(C_{\nabla^Z}(X, Y)U)W + \eta(W)(\phi C_{\nabla^Z})(X, Y)U. \end{aligned} \quad (4.11)$$

Now covariant differentiation of (4.8) with respect to W , yields

$$(\nabla_W C_{\nabla^Z})(X, Y)U = (\nabla_W C)(X, Y)U. \quad (4.12)$$

Making use of (4.8) and (4.12) in (4.11) we obtain

$$\begin{aligned} (\nabla_W^Z C_{\nabla^Z})(X, Y)U &= (\nabla_W C)(X, Y)U + g(W, C(X, Y)U)\xi \\ &\quad - \eta(C(X, Y)U)W + \eta(W)(\phi C)(X, Y)U. \end{aligned} \quad (4.13)$$

Taking account of (4.9), we write (4.13) as

$$\begin{aligned} &(\nabla_W^Z C_{\nabla^Z})(X, Y)U \\ &= (\nabla_W C)(X, Y)U + R'(X, Y, U, W)\xi + \eta(W)\phi R(X, Y)U \\ &\quad - \frac{r}{n(n-1)}\{g(Y, U)(g(X, W)\xi + \eta(W)\phi X) - g(X, U)(g(Y, W)\xi + \eta(W)\phi Y)\} \\ &\quad - \left[\frac{r}{n(n-1)} + 1\right]\{g(X, U)\eta(Y)W - g(Y, U)\eta(X)W\}. \end{aligned} \quad (4.14)$$

Applying ϕ^2 on both sides of above equation; then using (2.1) and (2.2) in (4.14) we have

$$\begin{aligned} &\phi^2(\nabla_W^Z C_{\nabla^Z})(X, Y)U \\ &= \phi^2(\nabla_W C)(X, Y)U + \eta(W)\phi R(X, Y)U \\ &\quad - \frac{r}{n(n-1)}\{g(Y, U)\phi X - g(X, U)\phi Y\}\eta(W) \\ &\quad - \left[\frac{r}{n(n-1)} + 1\right]\{g(X, U)\eta(Y) - g(Y, U)\eta(X)\}(W - \eta(W)\xi). \end{aligned} \quad (4.15)$$

If we consider X, Y, U, W orthogonal to ξ , (4.15) reduces to

$$\phi^2((\nabla_W^Z C_{\nabla^Z})(X, Y)U) = \phi^2((\nabla_W C)(X, Y)U). \quad (4.16)$$

This ends the proof of the required result. $\square$

Proposition 4.6. *Let M be an n -dimensional ($n > 1$) locally concircular ϕ -symmetric para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection ∇^Z . If the scalar curvature r with respect to the Levi-Civita connection ∇ is constant, then M is locally ϕ -symmetric.*

Proof. Now, from (4.9) we have

$$(\nabla_W C)(X, Y)U = (\nabla_W R_{\nabla})(X, Y)U - \frac{(\nabla_W r)}{n(n-1)}[g(Y, U)X - g(X, U)Y]. \quad (4.17)$$

From (4.17) in (4.16) we obtain

$$\phi^2((\nabla_W^Z C_{\nabla^Z})(X, Y)U) = \phi^2((\nabla_W R)(X, Y)U) - \frac{(\nabla_W r)}{n(n-1)}[g(Y, U)\phi^2 X - g(X, U)\phi^2 Y]. \quad (4.18)$$

By virtue of (2.1) in (4.18) and then taking X, Y, U, W orthogonal to ξ , we get

$$\phi^2(\nabla_W^Z C_{\nabla^Z})(X, Y)U = \phi^2((\nabla_W R)(X, Y)U) - \frac{(\nabla_W r)}{n(n-1)}[g(Y, U)X - g(X, U)Y]. \quad (4.19)$$

If r is constant, then $\nabla_W r$ is zero. Therefore, (4.19) gives

$$\phi^2(C_{\nabla^Z})(X, Y)U = \phi^2((\nabla_W R)(X, Y)U).$$

Hence, it completes the proof of the required result. $\square$

5 Local ϕ -Ricci symmetry with respect to the connections ∇^Z and ∇

Definition 5.1. *A para-Kenmotsu manifold M is said to be locally ϕ -Ricci symmetric with respect to the Zamkovoy canonical paracontact connection ∇^Z if its Ricci operator Q_{∇^Z} satisfies*

$$\phi^2((\nabla_W^Z Q_{\nabla^Z})X) = 0, \quad (5.1)$$

for any vector fields X, W orthogonal to ξ , and $S_{\nabla^Z}(X, W) = g(Q_{\nabla^Z} X, W)$.

Proposition 5.2. *If a para-Kenmotsu manifold is locally ϕ -Ricci symmetric with respect to the Zamkovoy canonical paracontact connection, then the manifold is Ricci symmetric.*

Proof. Let us consider a para-Kenmotsu manifold, which is locally ϕ -Ricci symmetric with respect to the connection ∇^Z . Then by virtue of (2.1) it follows from (5.1) that

$$(\nabla_W^Z Q_{\nabla^Z})X - \eta((\nabla_W^Z Q_{\nabla^Z})X)\xi = 0. \quad (5.2)$$

From (3.4) we can write

$$Q_{\nabla^Z}X = QX + (n-1)X. \quad (5.3)$$

Again we have

$$(\nabla_W^Z Q_{\nabla^Z})X = \nabla_W^Z Q_{\nabla^Z}X - Q_{\nabla^Z}(\nabla_W^Z X), \quad (5.4)$$

Using (5.3) in (5.4) we get

$$(\nabla_W^Z Q_{\nabla^Z})X = (\nabla_W^Z Q)X. \quad (5.5)$$

Taking account of (5.5), (5.2) reduces to

$$(\nabla_W^Z Q)X - \eta((\nabla_W^Z Q)X)\xi = 0. \quad (5.6)$$

From (3.2) it follows that,

$$\begin{aligned} (\nabla_W^Z Q)X &= \nabla_W^Z QX - Q(\nabla_W^Z X), \\ &= (\nabla_W Q)X + S(W, X)\xi + (n-1)(\eta(X)W + g(W, X)\xi) \\ &\quad + \eta(X)QW + \eta(W)(\phi QX - Q\phi X) \end{aligned} \quad (5.7)$$

and

$$\eta((\nabla_W^Z Q)X) = \eta((\nabla_W Q)X) + S(W, X) + (n-1)g(W, X). \quad (5.8)$$

Using (5.7) and (5.8) we get from (5.6) that

$$(\nabla_W Q)X + (n-1)\eta(X)W + \eta(W)(\phi QX - Q\phi X) + \eta(X)QW - \eta((\nabla_W Q)X)\xi = 0. \quad (5.9)$$

Taking inner product with U of (5.9) and considering X, W, U orthogonal to ξ , we get

$$(\nabla_W S)(X, U) = 0, \quad (5.10)$$

which implies that the manifold is Ricci symmetric with respect to the Levi-Civita connection ∇ .

Hence the proof. $\square$

Proposition 5.3. *A locally ϕ -Ricci symmetric para-Kenmotsu manifold with respect to the Zamkovoy canonical paracontact connection is an Einstein manifold.*

Proof. Putting $X = \xi$ in (5.10) and using (2.11), we get

$$S(W, U) = -(n-1)g(W, U), \quad (5.11)$$

for any vector fields $W, U \in \chi(M)$.

This ends the required proof. $\square$

6 A para-Kenmotsu manifold M whose curvature tensor is covariant constant with respect to the connection ∇^Z and M is recurrent with respect to the connection ∇

Definition 6.1. A para-Kenmotsu manifold M with respect to the Levi-Civita connection is said to be recurrent [11] if its curvature tensor R satisfies the condition.

$$(\nabla_W R)(X, Y)U = A(W)R(X, Y)U, \quad (6.1)$$

where A is a non-zero 1-form and $X, Y, U, W \in \chi(M)$.

Proposition 6.2. If in a para-Kenmotsu manifold the curvature tensor is covariant constant with respect to the Zamkovoy canonical paracontact connection and the manifold is recurrent with respect to the Levi-Civita connection, then the manifold is an η -Einstein manifold.

Proof. From (3.2), we can write (6.1) as

$$\begin{aligned} (\nabla_W^Z R)(X, Y)U &= \nabla_W^Z R(X, Y)U - R(\nabla_W^Z X, Y)U - R(X, \nabla_W^Z Y)U - R(X, Y)\nabla_W^Z U, \\ &= (\nabla_W R)(X, Y)U + g(W, R(X, Y)U)\xi - \eta(R(X, Y)U)W \\ &\quad + \eta(X)R(W, Y)U + \eta(Y)R(X, W)U + \eta(U)R(X, Y)W \\ &\quad + \eta(W)\{\phi R(X, Y)U - R(\phi X, Y)U - R(X, \phi Y)U - R(X, Y)\phi U\} \\ &\quad - g(X, W)R(\xi, Y)U - g(Y, W)R(X, \xi)U - g(U, W)R(X, Y)\xi. \end{aligned} \quad (6.2)$$

Using (2.7)-(2.9) in (6.2), we obtain

$$\begin{aligned} (\nabla_W^Z R)(X, Y)U &= (\nabla_W R)(X, Y)U + g(W, R(X, Y)U)\xi \\ &\quad + \eta(X)R(W, Y)U + \eta(Y)R(X, W)U + \eta(U)R(X, Y)W \\ &\quad + \eta(W)\{\phi R(X, Y)U - R(\phi X, Y)U - R(X, \phi Y)U - R(X, Y)\phi U\} \\ &\quad - g(X, U)\eta(Y)W + g(Y, U)\eta(X)W - g(X, W)\{\eta(U)Y - g(Y, U)\xi\} \\ &\quad - g(Y, W)\{g(X, U)\xi - \eta(U)X\} - g(W, U)\{\eta(X)Y - \eta(Y)X\}. \end{aligned} \quad (6.3)$$

Let $(\nabla_W^Z R)(X, Y)U = 0$, then from (6.3), it follows that

$$\begin{aligned} &(\nabla_W R)(X, Y)U + g(W, R(X, Y)U)\xi \\ &\quad + \eta(X)R(W, Y)U + \eta(Y)R(X, W)U + \eta(U)R(X, Y)W \\ &\quad + \eta(W)\{\phi R(X, Y)U - R(\phi X, Y)U - R(X, \phi Y)U - R(X, Y)\phi U\} \\ &\quad - g(X, U)\eta(Y)W + g(Y, U)\eta(X)W - g(X, W)\{\eta(U)Y - g(Y, U)\xi\} \\ &\quad - g(Y, W)\{g(X, U)\xi - \eta(U)X\} - g(W, U)\{\eta(X)Y - \eta(Y)X\} = 0. \end{aligned} \quad (6.4)$$

Now using (6.1) in (6.4), we have

$$\begin{aligned}
 & A(W)R(X, Y)U + g(W, R(X, Y)U)\xi \\
 & + \eta(X)R(W, Y)U + \eta(Y)R(X, W)U + \eta(U)R(X, Y)W \\
 & + \eta(W)\{\phi R(X, Y)U - R(\phi X, Y)U - R(X, \phi Y)U - R(X, Y)\phi U\} \\
 & - g(X, U)\eta(Y)W + g(Y, U)\eta(X)W - g(X, W)\{\eta(U)Y - g(Y, U)\xi\} \\
 & - g(Y, W)\{g(X, U)\xi - \eta(U)X\} - g(U, W)\{\eta(X)Y - \eta(Y)X\} = 0.
 \end{aligned} \tag{6.5}$$

Taking the inner product of (6.5) with ξ and using (2.2) and (2.9), it follows that

$$\begin{aligned}
 & A(W)\{g(X, U)\eta(Y) - g(Y, U)\eta(X)\} + g(W, R(X, Y)U) \\
 & + \eta(W)\{g(\phi Y, U)\eta(X) - g(\phi X, U)\eta(Y) - g(X, \phi U)\eta(Y) \\
 & + g(Y, \phi U)\eta(X)\} + g(X, W)g(Y, U) - g(Y, W)g(X, U) = 0.
 \end{aligned} \tag{6.6}$$

Contracting (6.6) over X and W , we obtain

$$S(Y, U) = \{A(\xi) - (n - 1)\}g(Y, U) - A(U)\eta(Y). \tag{6.7}$$

Since the Ricci tensor S with respect to the connection ∇ is symmetric; then from (6.7), we get

$$A(U)\eta(Y) = A(Y)\eta(U). \tag{6.8}$$

Putting $Y = \xi$ in (6.8) and using (2.2) we have

$$A(U) = A(\xi)\eta(U). \tag{6.9}$$

Combining (6.7) and (6.9), it follows that

$$S(Y, U) = \{A(\xi) - (n - 1)\}g(Y, U) - A(\xi)\eta(Y)\eta(U). \tag{6.10}$$

This results shows that the manifold is an η -Einstein manifold. Hence the proof. $\square$

7 Example

We consider the 3-dimensional manifold $M^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$ where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields (see [27], example of section 7)

$$X = \frac{\partial}{\partial x}, \quad \phi X = \frac{\partial}{\partial y}, \quad \xi = (x + 2y)\frac{\partial}{\partial x} + (2x + y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point of M^3 .

The 1-form $\eta = dz$ defines an almost paracontact structure on M^3 with characteristic vector field ξ . Let g, ϕ be the semi-Riemannian metric and the $(1, 1)$ tensor field given by

$$g = \begin{pmatrix} 1 & 0 & -(x+2y) \\ 0 & -1 & (2x+y) \\ -(x+2y) & (2x+y) & 1 - (2x+y)^2 + (x+2y)^2 \end{pmatrix}$$

$$\varphi = \begin{pmatrix} 0 & 1 & -(2x+y) \\ 1 & 0 & -(x+2y) \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. Clearly, (ϕ, ξ, η, g) defines an almost paracontact metric structure on M^3 . Let ∇ be the Levi-Civita connection with metric g , then we have

$$[X, \phi X] = 0, \quad [X, \xi] = X + 2\phi X, \quad [\phi X, \xi] = 2X + \phi X.$$

Next, by using the well-known Koszul's formula, we obtain

$$\begin{aligned} \nabla_X X &= -\xi, & \nabla_{\phi X} X &= 0, & \nabla_\xi X &= -2\phi X, \\ \nabla_X \phi X &= 0, & \nabla_{\phi X} \phi X &= \xi, & \nabla_\xi \phi X &= -2X, \\ \nabla_X \xi &= X, & \nabla_{\phi X} \xi &= \phi X, & \nabla_\xi \xi &= 0. \end{aligned}$$

Hence, from the above it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a para-Kenmotsu manifold. By the above results, one can easily compute

$$\begin{aligned} R(X, \phi X)\xi &= 0, & R(\phi X, \xi)\xi &= -\phi X, & R(X, \xi)\xi &= -X, \\ R(X, \phi X)\phi X &= X, & R(\phi X, \xi)\phi X &= -\xi, & R(X, \xi)\phi X &= 0, \\ R(X, \phi X)X &= \phi X, & R(\phi X, \xi)X &= 0, & R(X, \xi)X &= \xi. \end{aligned} \tag{7.1}$$

Using (7.1), we have constant scalar curvature as follows:

$$r = S(X, X) - S(\phi X, \phi X) + S(\xi, \xi) = -6.$$

Now consider the Zamkovoy canonical paracontact connection ∇^Z defined by (3.2) such that

$$\begin{aligned} \nabla_X^Z X &= 0, & \nabla_{\phi X}^Z X &= 0, & \nabla_\xi^Z X &= -\phi X, \\ \nabla_X^Z \phi X &= 0, & \nabla_{\phi X}^Z \phi X &= 0, & \nabla_\xi^Z \phi X &= -X, \\ \nabla_X^Z \xi &= 0, & \nabla_{\phi X}^Z \xi &= 0, & \nabla_\xi^Z \xi &= 0. \end{aligned}$$

Again, by the above results we can compute the components of curvature tensors with respect to the connection ∇^Z as follows:

$$\begin{aligned} R_{\nabla^Z}(X, \phi X)\xi &= 0, & R_{\nabla^Z}(\phi X, \xi)\xi &= 0, & R_{\nabla^Z}(X, \xi)\xi &= 0, \\ R_{\nabla^Z}(X, \phi X)\phi X &= 0, & R_{\nabla^Z}(\phi X, \xi)\phi X &= 0, & R_{\nabla^Z}(X, \xi)\phi X &= 0, \\ R_{\nabla^Z}(X, \phi X)X &= 0, & R_{\nabla^Z}(\phi X, \xi)X &= 0, & R_{\nabla^Z}(X, \xi)X &= 0. \end{aligned} \tag{7.2}$$

Using (7.2), we have constant scalar curvature r_{∇^z} as follows:

$$r_{\nabla^z} = S_{\nabla^z}(X, X) - S_{\nabla^z}(\phi X, \phi X) + S_{\nabla^z}(\xi, \xi) = 0.$$

The above arguments easily verifies all the properties of Remark 3.2 and Proposition 3.1.

Acknowledgement

Authors are grateful to the referees for their valuable suggestions in improvement of the paper.

References

- [1] A. M. Blaga, “ η -Ricci solitons on para-Kenmotsu manifolds”, *Balkan J. Geom. Appl.*, vol. 20, no. 1, pp. 1-13, 2015.
- [2] A. M. Blaga, “Canonical connections on para-Kenmotsu manifolds”, *Novi Sad J. Math.*, vol. 45, no. 2, pp. 131-142, 2015.
- [3] A. M. Blaga, “Generalized dual connections on para-Kenmotsu manifolds”, *Bull. Int. Math. Virtual Inst.*, vol. 7, no. 1, pp. 165-171, 2017.
- [4] A. M. Blaga, “Invariant, anti-invariant and slant submanifolds of a para-Kenmotsu manifold”, *Geom. Balkan Press, BSG Proc.* vol. 24, pp. 19-28, 2017.
- [5] U. C. De and A. Sarkar, “On ϕ -Ricci symmetric Sasakian manifolds”, *Proc. Jangjeon Math. Soc.*, vol. 11, no. 1, pp. 47-52, 2008.
- [6] U. C. De, A. A. Shaikh and S. Biswas, “On ϕ -recurrent Sasakian manifolds”, *Novi Sad J. Math.*, vol. 33, no. 2, pp. 43-48, 2003.
- [7] U. C. De, A. Yildiz and A. F. Yaliniz, “On ϕ -recurrent Kenmotsu manifolds”, *Turk. J. Math.*, vol. 33, pp. 17-25, 2009.
- [8] S. Ghosh and U. C. De, “On ϕ -Ricci symmetric (κ, μ) -contact metric manifolds”, *Acta Math. Univ. Comenian. (N.S.)*, vol. 86, no. 2 pp. 205-213, 2017.
- [9] S. Kaneyuki, F. L. Williams, “Almost paracontact and parahodge structures on manifolds”, *Nagoya Math. J.*, vol. 99, pp. 173-187, 1985.
- [10] K. Kenmotsu, “A class of almost contact Riemannian manifolds”, *Tohoku Math. J.*, vol. 24, pp. 93-103, 1972.
- [11] E. M. Patterson, “Some theorems on Ricci-recurrent spaces”, *journal London Math. Soc.*, vol. 27, pp. 287-295, 1952.
- [12] D. G. Prakasha and K. K. Mirji, “On ϕ -symmetric $N(\kappa)$ -paracontact metric manifolds”, *J. Math.*, Vol. 2015, Article ID 728298, 6 pages, 2015.
- [13] D. G. Prakasha and K. Vikas, “On ϕ -recurrent para-Kenmotsu manifolds”, *Int. J. Pure & Engg. Mathematics*, vol. 3, no. II, pp. 17-26, 2015.
- [14] D. G. Prakasha and B. S. Hadimani, “On Generalized recurrent para-Kenmotsu manifolds”, *Glob. J. Pure Appl. Math.*, vol 11, no. 2, pp. 1049-1059, 2015.
- [15] K. L. Sai Prasad, T. Satyanarayana, “On para-Kenmotsu manifold”, *Int. J. Pure Appl. Math.*, vol. 90, no. 1, pp. 35-41, 2014.

- [16] T. Satyanarayana, K. L. Sai Prasad, B. Satyanarayana, “On a class of para Kenmotsu manifolds”, *Int. J. Pure Appl. Math.*, vol. 115, no. 4), pp. 827-834, 2017.
- [17] A. A. Shaikh and K. K. Baishya, “On ϕ -symmetric LP-Sasakian manifolds”, *Yokohama Math. J.*, vol. 52, pp. 97-112, 2005.
- [18] A. A. Shaikh, T. Basu and S. Eyasmin, “On locally ϕ -symmetric $(LCS)_n$ -manifolds”, *Int. J. Pure Appl. Math.*, vol. 41, no. 8, pp. 1161-1170, 2007.
- [19] S. S. Shukla and M. K. Shukla, “On ϕ -symmetric para-Sasakian manifolds”, *Int. J. Math. Anal.*, vol. 4, no. 16, pp. 761-769, 2010.
- [20] B. B. Sinha, K. L. Sai Prasad, “A class of Almost paracontact metric manifold”, *Bull. Calcutta Math. Soc.*, vol. 87, pp. 307-312, 1995.
- [21] T. Takahashi, “Sasakian ϕ -Symmetric Spaces”, *Tohoku Math. J.*, vol. 29, pp. 91-113, 1977.
- [22] S. Tanno, “Variational problems on contact Riemannian manifolds”, *Trans. Amer. Math. Soc.*, vol. 314, no. 1, pp. 349-379, 1989.
- [23] J. Welyczko, “Slant curves in 3-dimensional normal almost paracontact metric manifolds”, *Mediterr. J. Math.*, 2013.
- [24] K. Yano, “Concircular geometry I, Concircular transformations”, *Proc. Imp. Acad. Tokyo*, vol. 16, no. 16, pp. 195-200, 1940.
- [25] S. Zamkovoy, “Canonical connections on paracontact manifolds”, *Ann. Glob. Anal. Geom.*, vol. 36, no. 1, pp. 37-60, 2009.
- [26] S. Zamkovoy, “On para-Kenmotsu manifolds”, *Filomat*, vol. 32, no. 14, pp. 4971-4980, 2018.
- [27] S. Zamkovoy, G. Nakova, “The decomposition of almost paracontact metric manifolds in eleven classes revisited”, *J. Geom.*, vol. 109, no. 18, 23 pages, 2018.