Results on para-Sasakian manifold admitting a quarter symmetric metric connection

VISHNUVARDHANA. S.V.¹ AND VENKATESHA²

¹ *Department of Mathematics, GITAM School of Science, GITAM (Deemed to be University)
Bengaluru, Karnataka-561 203, INDIA.*

² *Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga,
Karnataka, INDIA.*

svvishnuvardhana@gmail.com, vensmath@gmail.com

ABSTRACT

In this paper we have studied pseudosymmetric, Ricci-pseudosymmetric and projectively pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection and constructed examples of 3-dimensional and 5-dimensional para-Sasakian manifold admitting a quarter-symmetric metric connection to verify our results.

RESUMEN

En este artículo hemos estudiado variedades para-Sasakianas seudosimétricas, Ricci-seudosimétricas y proyectivamente seudosimétricas que admiten una conexión métrica cuarto-simétrica, y construimos ejemplos de variedades para-Sasakianas 3-dimensional y 5-dimensional que admiten una conexión métrica cuarto-simétrica para verificar nuestros resultados.

Keywords and Phrases: Para-Sasakian manifold, pseudosymmetric, Ricci-pseudosymmetric, projectively pseudosymmetric, quarter-symmetric metric connection.

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1 Introduction

One of the most important geometric property of a space is symmetry. Spaces admitting some sense of symmetry play an important role in differential geometry and general relativity. Cartan [5] introduced locally symmetric spaces, i.e., the Riemannian manifold (M, g) for which $\nabla R = 0$, where ∇ denotes the Levi-Civita connection of the metric. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$. Thus, every locally symmetric space satisfies $R \cdot R = 0$, whereby the first R stands for the curvature operator of (M, g) , i.e., for tangent vector fields X and Y one has $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$, which acts as a derivation on the second R which stands for the Riemann-Christoffel curvature tensor. The converse however does not hold in general. The spaces for which $R \cdot R = 0$ holds at every point were called semi-symmetric spaces and which were classified by Szabo [19].

Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. In some spaces $R \cdot R$ is not identically zero, these turn out to be the pseudo-symmetric spaces of Deszcz [9, 10, 11], which were characterized by the condition $R \cdot R = L Q(g, R)$, where L is a real function on M and $Q(g, R)$ is the Tachibana tensor of M .

If at every point of M the curvature tensor satisfies the condition

$$R(X, Y) \cdot \mathcal{J} = L_{\mathcal{J}}[(X \wedge_g Y) \cdot \mathcal{J}], \quad (1.1)$$

then a Riemannian manifold M is called pseudosymmetric (resp., Ricci-pseudosymmetric, projectively pseudosymmetric) when $\mathcal{J} = R(\text{resp.}, S, P)$. Here $(X \wedge_g Y)$ is an endomorphism and is defined by $(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y$ and $L_{\mathcal{J}}$ is some function on $U_{\mathcal{J}} = \{x \in M : \mathcal{J} \neq 0\}$ at x . A geometric interpretation of the notion of pseudosymmetry is given in [13]. It is also easy to see that every pseudosymmetric manifold is Ricci-pseudosymmetric, but the converse is not true.

An analogue to the almost contact structure, the notion of almost paracontact structure was introduced by Sato [18]. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be of even dimension as well. Kaneyuki and Williams [14] studied the almost paracontact structure on a pseudo-Riemannian manifold. Recently, almost paracontact geometry in particular, para-Sasakian geometry has taking interest, because of its interplay with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics ([4, 7, 8], etc.).

As a generalization of semi-symmetric connection, quarter-symmetric connection was introduced. Quarter-symmetric connection on a differentiable manifold with affine connection was defined and studied by Golab [12]. From thereafter many geometers studied this connection on different manifolds.

Para-Sasakian manifold with respect to quarter-symmetric metric connection was studied by

De et.al., [16, 1], Pradeep Kumar et.al., [17] and Bisht and Shanker [15].

Motivated by the above studies in this article we study properties of projective curvature tensor on para-Sasakian manifold admitting a quarter-symmetric metric connection. The organization of the paper is as follows: In Section 2, we present some basic notions of para-Sasakian manifold and quarter-symmetric metric connection on it. Section 3 and 4 are respectively devoted to study the pseudosymmetric and Ricci-pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection. Here we prove that if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Pseudosymmetric (resp., Ricci pseudosymmetric) then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\bar{R}} = -2$ (resp., $L_{\bar{S}} = -2$). Section 5 and 6 are concerned with projectively flat and projectively pseudosymmetric para-Sasakian manifold M^n admitting a quarter-symmetric metric connection. Finally, we construct examples of 3-dimensional and 5-dimensional para-Sasakian manifold admitting a quarter-symmetric metric connection and we find some of its geometric characteristics.

2 Preliminaries

A differential manifold M^n is said to admit an almost paracontact Riemannian structure (ϕ, ξ, η, g) , where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M^n such that

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad (2.1)$$

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

for all vector fields $X, Y \in \chi(M^n)$. If (ϕ, ξ, η, g) on M^n satisfies the following equations

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.3)$$

$$d\eta = 0 \quad \text{and} \quad \nabla_X \xi = \phi X, \quad (2.4)$$

then M^n is called para-Sasakian manifold [3].

In a para-Sasakian manifold, the following relations hold [6]:

$$(\nabla_X \eta)Y = -g(X, Y) + \eta(X)\eta(Y), \quad (2.5)$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (2.6)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad (2.7)$$

$$S(X, \xi) = -(n-1)\eta(X), \quad (2.8)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \quad (2.9)$$

for every vector fields X, Y, Z on M^n . Here ∇ denotes the Levi-Civita connection, R denotes the Riemannian curvature tensor and S denotes the Ricci curvature tensor.

Here we consider a quarter-symmetric metric connection $\tilde{\nabla}$ on a para-Sasakian manifold [16] given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \quad (2.10)$$

The relation between curvature tensor $\tilde{R}(X, Y)Z$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ and the curvature tensor $R(X, Y)Z$ with respect to the Levi-Civita connection ∇ is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + 3g(\phi X, Z)\phi Y - 3g(\phi Y, Z)\phi X \\ &+ \{\eta(X)Y - \eta(Y)X\}\eta(Z) - [g(Y, Z)\eta(X) - \eta(Y)g(X, Z)]\xi. \end{aligned} \quad (2.11)$$

Also from (2.11) we obtain

$$\tilde{S}(Y, Z) = S(Y, Z) + 2g(Y, Z) - (n+1)\eta(Y)\eta(Z) - 3\text{trace}\phi g(\phi Y, Z), \quad (2.12)$$

where $\tilde{S}$ and S are Ricci tensors of connections $\tilde{\nabla}$ and ∇ respectively.

3 Pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is said to be pseudosymmetric if

$$\tilde{R}(X, Y) \cdot \tilde{R} = L_{\tilde{R}}[(X \wedge_g Y) \cdot \tilde{R}], \quad (3.1)$$

holds on the set $U_{\tilde{R}} = \{x \in M^n : \tilde{R} \neq 0 \text{ at } x\}$, where $L_{\tilde{R}}$ is some function on $U_{\tilde{R}}$.

Suppose that M^n be pseudosymmetric, then in view of (3.1) we have

$$\begin{aligned} &\tilde{R}(\xi, Y)\tilde{R}(U, V)W - \tilde{R}(\tilde{R}(\xi, Y)U, V)W - \tilde{R}(U, \tilde{R}(\xi, Y)V)W \\ &- \tilde{R}(U, V)\tilde{R}(\xi, Y)W = L_{\tilde{R}}[(\xi \wedge_g Y)\tilde{R}(U, V)W - \tilde{R}((\xi \wedge_g Y)U, V)W \\ &- \tilde{R}(U, (\xi \wedge_g Y)V)W - \tilde{R}(U, V)(\xi \wedge_g Y)W]. \end{aligned} \quad (3.2)$$

By virtue of (2.7) and (2.11), (3.2) takes the form

$$\begin{aligned} &(L_{\tilde{R}} + 2)[\eta(\tilde{R}(U, V)W)Y - g(Y, \tilde{R}(U, V)W)\xi - \eta(U)\tilde{R}(Y, V)W + g(Y, U)\tilde{R}(\xi, V)W \\ &- \eta(V)\tilde{R}(U, Y)W + g(Y, V)\tilde{R}(U, \xi)W - \eta(W)\tilde{R}(U, V)Y + g(Y, W)\tilde{R}(U, V)\xi] = 0. \end{aligned} \quad (3.3)$$

Taking inner product of (3.3) with ξ and using (2.6) and (2.11), we get

$$\begin{aligned} &(L_{\tilde{R}} + 2)[g(Y, R(U, V)W) + 3g(\phi U, W)g(\phi V, Y) - 3g(\phi V, W)g(\phi U, Y) \\ &+ \eta(W)\{\eta(U)g(V, Y) - \eta(V)g(U, Y)\} - \{g(V, W)\eta(U) - \eta(V)g(U, W)\}\eta(Y) \\ &+ 2\{g(V, W)g(Y, U) - g(V, Y)g(U, W)\}] = 0. \end{aligned} \quad (3.4)$$

Assuming that $L_{\tilde{R}} + 2 \neq 0$, the above equation becomes

$$\begin{aligned} &g(Y, R(U, V)W) + 3g(\phi U, W)g(\phi V, Y) - 3g(\phi V, W)g(\phi U, Y) \\ &+ \eta(W)\{\eta(U)g(V, Y) - \eta(V)g(U, Y)\} - [g(V, W)\eta(U) - \eta(V)g(U, W)]\eta(Y) \\ &+ 2[g(V, W)g(Y, U) - g((V, Y)g(U, W))] = 0. \end{aligned} \quad (3.5)$$

Putting $V = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i , $i = 1, 2, 3, \dots, n$, we get

$$\tilde{S}(Y, U) = -2(n-1)g(Y, U). \quad (3.6)$$

Hence, we can state the following:

Theorem 1. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{R}} = -2$.*

4 Ricci-pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is said to be Ricci-pseudosymmetric if the following condition is satisfied

$$\tilde{R}(X, Y) \cdot \tilde{S} = L_{\tilde{S}}[(X \wedge_g Y) \cdot \tilde{S}], \quad (4.1)$$

on $U_{\tilde{S}}$.

Let para-Sasakian manifold M^n admitting a quarter-symmetric metric connection be Ricci-pseudosymmetric. Then we have

$$\tilde{S}(\tilde{R}(X, Y)Z, W) + \tilde{S}(Z, \tilde{R}(X, Y)W) = L_{\tilde{S}}[\tilde{S}((X \wedge_g Y)Z, W) + \tilde{S}(Z, (X \wedge_g Y)W)]. \quad (4.2)$$

By taking $Y = W = \xi$ and making use of (2.7), (2.8) and (2.11), the above equation turns into

$$(L_{\tilde{S}} + 2)[\tilde{S}(X, Z) + 2(n-1)g(X, Z)] = 0 \quad (4.3)$$

Thus, we have the following assertion:

Theorem 2. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci-pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{S}} = -2$.*

5 Projectively flat para-Sasakian manifold admitting a quarter-symmetric metric connection

The projective curvature tensor on a Riemannian manifold is defined by [2]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[S(Y, Z)X - S(X, Z)Y]. \quad (5.1)$$

For an n -dimensional para-Sasakian manifold M^n admitting a quarter-symmetric metric connection, the projective curvature tensor is given by

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{(n-1)}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y]. \quad (5.2)$$

Theorem 3. *A projectively flat para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is an Einstein manifold with respect to quarter-symmetric metric connection.*

Proof. Consider a projectively flat para-Sasakian manifold admitting a quarter-symmetric metric connection. Then from (5.2) we have

$$g(\tilde{R}(X, Y)Z, W) = \frac{1}{(n-1)}[\tilde{S}(Y, Z)g(X, W) - \tilde{S}(X, Z)g(Y, W)]. \quad (5.3)$$

Setting $X = W = \xi$ in (5.3) and using (2.7), (2.8), (2.11) and (2.12), we get

$$\tilde{S}(X, Z) = -2(n-1)g(X, Z). \quad (5.4)$$

Hence, the proof is completed. $\square$

6 Projectively pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold admitting a quarter-symmetric metric connection is said to be projectively pseudosymmetric if

$$\tilde{R}(X, Y) \cdot \tilde{P} = L_{\tilde{P}}[(X \wedge_g Y) \cdot \tilde{P}], \quad (6.1)$$

holds on the set $U_{\tilde{P}} = \{x \in M^n : \tilde{P} \neq 0 \text{ at } x\}$, where $L_{\tilde{P}}$ is some function on $U_{\tilde{P}}$.

Let M^n be projectively pseudosymmetric, then we have

$$\begin{aligned} & \tilde{R}(X, \xi)\tilde{P}(U, V)\xi - \tilde{P}(\tilde{R}(X, \xi)U, V)\xi - \tilde{P}(U, \tilde{R}(X, \xi)V)\xi \\ & - \tilde{P}(U, V)\tilde{R}(X, \xi)\xi = L_{\tilde{P}}[(X \wedge_g \xi)\tilde{P}(U, V)\xi - \tilde{P}((X \wedge_g \xi)U, V)\xi \\ & - \tilde{P}(U, (X \wedge_g \xi)V)\xi - \tilde{P}(U, V)(X \wedge_g \xi)\xi]. \end{aligned} \quad (6.2)$$

By virtue of (2.11), (2.12) and (5.2), (6.2) becomes

$$(L_{\tilde{P}} + 2)\tilde{P}(U, V)X = 0. \quad (6.3)$$

So, one can state that:

Theorem 4. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is projectively pseudosymmetric then M^n is projectively flat with respect to quarter-symmetric metric connection or $L_{\tilde{P}} = -2$.*

In view of theorem 3, one can state the above theorem as

Theorem 5. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is projectively pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or $L_{\tilde{P}} = -2$.*

7 Examples

7.1 Example

We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame field on M given by

$$E_1 = e^z \frac{\partial}{\partial y}, \quad E_2 = e^z \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right), \quad E_3 = \frac{\partial}{\partial z},$$

If g is a Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \leq i, j \leq 3$, and if η is the 1-form defined by $\eta(Z) = g(Z, E_3)$ for any vector field $Z \in \chi(M)$.

We define the $(1, 1)$ -tensor field ϕ as

$$\phi(E_1) = E_1, \quad \phi(E_2) = -E_2, \quad \phi(E_3) = 0.$$

The linearity property of ϕ and g yields that

$$\begin{aligned} \eta(E_3) &= 1, \\ \phi^2 U &= U - \eta(U)E_3, \\ g(\phi U, \phi V) &= g(U, V) - \eta(U)\eta(V), \end{aligned}$$

for any $U, V \in \chi(M)$.

Now we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.$$

The Riemannian connection ∇ of the metric g known as Koszul's formula and is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula we get the followings in matrix form

$$\begin{pmatrix} \nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 \\ \nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 \\ \nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & -E_3 & E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly (ϕ, ξ, η, g) is a para-Sasakian structure on M . Thus $M(\phi, \xi, \eta, g)$ is a 3-dimensional para-Sasakian manifold.

Using (2.10) and the above equation, one can easily obtain the following:

$$\begin{pmatrix} \tilde{\nabla}_{E_1} E_1 & \tilde{\nabla}_{E_1} E_2 & \tilde{\nabla}_{E_1} E_3 \\ \tilde{\nabla}_{E_2} E_1 & \tilde{\nabla}_{E_2} E_2 & \tilde{\nabla}_{E_2} E_3 \\ \tilde{\nabla}_{E_3} E_1 & \tilde{\nabla}_{E_3} E_2 & \tilde{\nabla}_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -2E_3 & 0 & 2E_1 \\ 0 & -2E_3 & 2E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

With the help of the above results it can be easily verified that

$$\begin{aligned} R(E_1, E_2)E_3 &= 0, & R(E_2, E_3)E_3 &= -E_2, & R(E_1, E_3)E_3 &= -E_1, \\ R(E_1, E_2)E_2 &= -E_1, & R(E_2, E_3)E_2 &= E_3, & R(E_1, E_3)E_2 &= 0, \\ R(E_1, E_2)E_1 &= E_2, & R(E_2, E_3)E_1 &= 0, & R(E_1, E_3)E_1 &= E_3. \end{aligned}$$

and

$$\begin{aligned} \tilde{R}(E_1, E_2)E_3 &= 0, & \tilde{R}(E_2, E_3)E_3 &= -2E_2, & \tilde{R}(E_1, E_3)E_3 &= -2E_1, \\ \tilde{R}(E_1, E_2)E_2 &= -4E_1, & \tilde{R}(E_2, E_3)E_2 &= 2E_3, & \tilde{R}(E_1, E_3)E_2 &= 0, \\ \tilde{R}(E_1, E_2)E_1 &= 4E_2, & \tilde{R}(E_2, E_3)E_1 &= 0, & \tilde{R}(E_1, E_3)E_1 &= 2E_3. \end{aligned} \quad (7.1)$$

Since E_1, E_2, E_3 forms a basis, any vector field $X, Y, Z \in \chi(M)$ can be written as $X = a_1 E_1 + b_1 E_2 + c_1 E_3$, $Y = a_2 E_1 + b_2 E_2 + c_2 E_3$, $Z = a_3 E_1 + b_3 E_2 + c_3 E_3$, where $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, 2, 3$. Using the expressions of the curvature tensors, we find values of Riemannian curvature

and Ricci curvature with respect to quarter-symmetric metric connection as;

$$\begin{aligned}\tilde{R}(X, Y)Z &= [-4\{a_1b_2 - b_1a_2\}b_3 + 2\{c_1a_2 - a_1c_2\}c_3]E_1 \\ &+ [-4\{b_1a_2 - a_1b_2\}a_3 + 2\{c_1b_2 - b_1c_2\}c_3]E_2 \\ &+ [-2\{c_1a_2 - a_1c_2\}a_3 - 2\{c_1b_2 - b_1c_2\}b_3]E_3,\end{aligned}\quad (7.2)$$

$$\tilde{S}(E_1, E_1) = \tilde{S}(E_2, E_2) = -6, \quad \tilde{S}(E_3, E_3) = -4. \quad (7.3)$$

Using (7.1), (7.3) and the expression of the endomorphism $(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y$, one can easily verify that

$$\tilde{S}(\tilde{R}(X, E_3)Y, E_3) + \tilde{S}(Y, \tilde{R}(X, E_3)E_3) = -2[\tilde{S}((X \wedge_g E_3)Y, E_3) + \tilde{S}(Y, (X \wedge_g E_3)E_3)], \quad (7.4)$$

here $L_{\tilde{S}} = -2$. Thus, the above equation verify one part of the Theorem 2 of section 4.

Moreover, the manifold under consideration satisfies

$$\begin{aligned}\tilde{R}(X, Y)Z &= -\tilde{R}(Y, X)Z, \\ \tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y &= 0.\end{aligned}$$

Hence, from the above equations one can say that this example verifies the condition (c) of Theorem 3.1 in [1] and first Bianchi identity.

7.2 Example

We consider a 5-dimensional manifold $M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5\}$, where $(x_1, x_2, x_3, x_4, x_5)$ are standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$E_1 = \frac{\partial}{\partial x_1}, \quad E_2 = \frac{\partial}{\partial x_2}, \quad E_3 = \frac{\partial}{\partial x_3}, \quad E_4 = \frac{\partial}{\partial x_4}, \quad E_5 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5},$$

which are linearly independent at each point of M .

Let g be a Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \leq i, j \leq 5$, and if η is the 1-form defined by $\eta(Z) = g(Z, E_5)$ for any vector field $Z \in \chi(M)$. Let ϕ be the $(1, 1)$ -tensor field defined by

$$\phi(E_1) = E_1, \quad \phi(E_2) = E_2, \quad \phi(E_3) = E_3, \quad \phi(E_4) = E_4, \quad \phi(E_5) = 0.$$

The linearity property of ϕ and g yields that

$$\begin{aligned}\eta(E_5) &= 1, \\ \phi^2 U &= U - \eta(U)E_5, \\ g(\phi U, \phi V) &= g(U, V) - \eta(U)\eta(V),\end{aligned}$$

for any $U, V \in \chi(M)$.

Now we have

$$\begin{aligned}[E_1, E_2] &= 0, & [E_1, E_3] &= 0, & [E_1, E_4] &= 0, & [E_1, E_5] &= E_1, \\ [E_2, E_3] &= 0, & [E_2, E_4] &= 0, & [E_2, E_5] &= E_2, \\ [E_3, E_4] &= 0, & [E_3, E_5] &= E_3, & [E_4, E_5] &= E_4.\end{aligned}$$

By virtue of Koszul's formula we get the followings in matrix form

$$\begin{pmatrix} \nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 & \nabla_{E_1} E_4 & \nabla_{E_1} E_5 \\ \nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 & \nabla_{E_2} E_4 & \nabla_{E_2} E_5 \\ \nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 & \nabla_{E_3} E_4 & \nabla_{E_3} E_5 \\ \nabla_{E_4} E_1 & \nabla_{E_4} E_2 & \nabla_{E_4} E_3 & \nabla_{E_4} E_4 & \nabla_{E_4} E_5 \\ \nabla_{E_5} E_1 & \nabla_{E_5} E_2 & \nabla_{E_5} E_3 & \nabla_{E_5} E_4 & \nabla_{E_5} E_5 \end{pmatrix} = \begin{pmatrix} -E_5 & 0 & 0 & 0 & E_1 \\ 0 & -E_5 & 0 & 0 & E_2 \\ 0 & 0 & -E_5 & 0 & E_3 \\ 0 & 0 & 0 & -E_5 & E_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Above expressions satisfies all the properties of para-Sasakian manifold. Thus $M(\phi, \xi, \eta, g)$ is a 5-dimensional para-Sasakian manifold.

From the above expressions and the relation of quarter symmetric metric connection and Riemannian connection, one can easily obtain the following:

$$\begin{pmatrix} \tilde{\nabla}_{E_1} E_1 & \tilde{\nabla}_{E_1} E_2 & \tilde{\nabla}_{E_1} E_3 & \tilde{\nabla}_{E_1} E_4 & \tilde{\nabla}_{E_1} E_5 \\ \tilde{\nabla}_{E_2} E_1 & \tilde{\nabla}_{E_2} E_2 & \tilde{\nabla}_{E_2} E_3 & \tilde{\nabla}_{E_2} E_4 & \tilde{\nabla}_{E_2} E_5 \\ \tilde{\nabla}_{E_3} E_1 & \tilde{\nabla}_{E_3} E_2 & \tilde{\nabla}_{E_3} E_3 & \tilde{\nabla}_{E_3} E_4 & \tilde{\nabla}_{E_3} E_5 \\ \tilde{\nabla}_{E_4} E_1 & \tilde{\nabla}_{E_4} E_2 & \tilde{\nabla}_{E_4} E_3 & \tilde{\nabla}_{E_4} E_4 & \tilde{\nabla}_{E_4} E_5 \\ \tilde{\nabla}_{E_5} E_1 & \tilde{\nabla}_{E_5} E_2 & \tilde{\nabla}_{E_5} E_3 & \tilde{\nabla}_{E_5} E_4 & \tilde{\nabla}_{E_5} E_5 \end{pmatrix} = \begin{pmatrix} -2E_5 & 0 & 0 & 0 & 2E_1 \\ 0 & -2E_5 & 0 & 0 & 2E_2 \\ 0 & 0 & -2E_5 & 0 & 2E_3 \\ 0 & 0 & 0 & -2E_5 & 2E_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

With the help of the above results it can be easily obtain the non-zero components of curvature tensors as

$$\begin{aligned}R(E_1, E_2)E_1 &= E_2, & R(E_1, E_2)E_2 &= -E_1, & R(E_1, E_3)E_1 &= E_3, & R(E_1, E_3)E_3 &= -E_1, \\ R(E_1, E_4)E_1 &= E_4, & R(E_1, E_4)E_4 &= -E_1, & R(E_1, E_5)E_1 &= E_5, & R(E_1, E_5)E_5 &= -E_1, \\ R(E_2, E_3)E_2 &= E_3, & R(E_2, E_3)E_3 &= -E_2, & R(E_2, E_4)E_2 &= E_4, & R(E_2, E_4)E_4 &= -E_2, \\ R(E_2, E_5)E_2 &= E_5, & R(E_2, E_5)E_5 &= -E_2, & R(E_3, E_4)E_3 &= E_4, & R(E_3, E_4)E_4 &= -E_3, \\ R(E_3, E_5)E_3 &= E_5, & R(E_3, E_5)E_5 &= -E_3, & R(E_4, E_5)E_4 &= E_5, & R(E_4, E_5)E_5 &= -E_4,\end{aligned}$$

and

$$\begin{aligned}\tilde{R}(E_1, E_2)E_1 &= 4E_2, & \tilde{R}(E_1, E_2)E_2 &= -4E_1, & \tilde{R}(E_1, E_3)E_1 &= 4E_3, & \tilde{R}(E_1, E_3)E_3 &= -4E_1, \\ \tilde{R}(E_1, E_4)E_1 &= 4E_4, & \tilde{R}(E_1, E_4)E_4 &= -4E_1, & \tilde{R}(E_1, E_5)E_1 &= 2E_5, & \tilde{R}(E_1, E_5)E_5 &= -2E_1, \\ \tilde{R}(E_2, E_3)E_2 &= 4E_3, & \tilde{R}(E_2, E_3)E_3 &= -4E_2, & \tilde{R}(E_2, E_4)E_2 &= 4E_4, & \tilde{R}(E_2, E_4)E_4 &= -4E_2, \\ \tilde{R}(E_2, E_5)E_2 &= 2E_5, & \tilde{R}(E_2, E_5)E_5 &= -2E_2, & \tilde{R}(E_3, E_4)E_3 &= 4E_4, & \tilde{R}(E_3, E_4)E_4 &= -4E_3, \\ \tilde{R}(E_3, E_5)E_3 &= 2E_5, & \tilde{R}(E_3, E_5)E_5 &= -2E_3, & \tilde{R}(E_4, E_5)E_4 &= 2E_5, & \tilde{R}(E_4, E_5)E_5 &= -2E_4.\end{aligned}\quad (7.5)$$

Since E_1, E_2, E_3, E_4, E_5 forms a basis, any vector field $X, Y, Z \in \chi(M)$ can be written as $X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4 + f_1E_5$, $Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4 + f_2E_5$, $Z = a_3E_1 + b_3E_2 + c_3E_3 + d_3E_4 + f_3E_5$, where $a_i, b_i, c_i, d_i, f_i \in \mathbb{R}$, $i = 1, 2, 3, 4, 5$. Using the expressions of the curvature tensors, we find values of Riemannian curvature and Ricci curvature with respect to quarter-symmetric metric connection as;

$$\begin{aligned}\tilde{R}(X, Y)Z &= [-4\{a_1(b_2b_3 + c_2c_3 + d_2d_3) - a_2(b_1b_3 + c_1c_3 + d_1d_3)\} - 2(a_1f_2 - f_1a_2)f_3]E_1 \\ &+ [-4\{b_1(a_2a_3 + c_2c_3 + d_2d_3) - b_2(a_1a_3 + c_1c_3 + d_1d_3)\} - 2(b_1f_2 - f_1b_2)f_3]E_2 \\ &+ [-4\{c_1(a_2a_3 + b_2b_3 + d_2d_3) - c_2(a_1a_3 + b_1b_3 + d_1d_3)\} - 2(c_1f_2 - f_1c_2)f_3]E_3 \\ &+ [-4\{d_1(a_2a_3 + b_2b_3 + c_2c_3) - d_2(a_1a_3 + b_1b_3 + c_1c_3)\} - 2(d_1f_2 - f_1d_2)f_3]E_4 \\ &+ [2\{(a_1f_2 - f_1a_2)a_3 + (b_1f_2 - f_1b_2)b_3 + (c_1f_2 - f_1c_2)c_3 + (d_1f_2 - f_1d_2)d_3\}]E_5, \\ \tilde{S}(E_1, E_1) &= \tilde{S}(E_2, E_2) = \tilde{S}(E_3, E_3) = \tilde{S}(E_4, E_4) = -14, \quad \tilde{S}(E_5, E_5) = -8.\end{aligned}\quad (7.6)$$

In view of (7.5), (7.6) and the expression of the endomorphism one can easily verify the equation (7.4) and hence the Theorem 2 of section 4 is verified. This example also verifies the condition (c) of Theorem 3.1 in [1] and first Bianchi identity.

Above two examples verifies the one part of the Theorem 2, that is, if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci pseudosymmetric then M^n satisfies $L_{\tilde{S}} = -2$ (M^n is not Einstein manifold with respect to quarter-symmetric metric connection). Another part of the theorem is that, if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection ($L_{\tilde{S}} \neq -2$). Now, the second part of the Theorem 2 can be verified by using the proper example.

7.3 Example

We consider a 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are standard coordinates in $\mathbb{R}^5$. Let $\{E_1, E_2, E_3, E_4, E_5\}$ be a linearly independent global frame field on M given

by

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = e^{-x} \frac{\partial}{\partial y}, \quad E_3 = e^{-x} \frac{\partial}{\partial z}, \quad E_4 = e^{-x} \frac{\partial}{\partial u}, \quad E_5 = e^{-x} \frac{\partial}{\partial v}.$$

Let g be a Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \leq i, j \leq 5$, and if η is the 1-form defined by $\eta(Z) = g(Z, E_1)$ for any vector field $Z \in \chi(M)$.

Let the $(1, 1)$ -tensor field ϕ be defined by

$$\phi(E_1) = 0, \quad \phi(E_2) = E_2, \quad \phi(E_3) = E_3, \quad \phi(E_4) = E_4, \quad \phi(E_5) = E_5.$$

With the help of linearity property of ϕ and g , we have

$$\begin{aligned} \eta(E_1) &= 1, \\ \phi^2 V &= V - \eta(V)E_1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

for any $X, Y \in \chi(M)$.

Now we have

$$\begin{aligned} [E_1, E_2] &= -E_2, \quad [E_1, E_3] = -E_3, \quad [E_1, E_4] = -E_4, \quad [E_1, E_5] = -E_5, \\ [E_2, E_3] &= [E_2, E_4] = [E_2, E_5] = [E_3, E_4] = [E_3, E_5] = [E_4, E_5] = 0. \end{aligned}$$

With the help of Koszul's formula we get the followings in matrix form

$$\begin{pmatrix} \nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 & \nabla_{E_1} E_4 & \nabla_{E_1} E_5 \\ \nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 & \nabla_{E_2} E_4 & \nabla_{E_2} E_5 \\ \nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 & \nabla_{E_3} E_4 & \nabla_{E_3} E_5 \\ \nabla_{E_4} E_1 & \nabla_{E_4} E_2 & \nabla_{E_4} E_3 & \nabla_{E_4} E_4 & \nabla_{E_4} E_5 \\ \nabla_{E_5} E_1 & \nabla_{E_5} E_2 & \nabla_{E_5} E_3 & \nabla_{E_5} E_4 & \nabla_{E_5} E_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ E_2 & -E_1 & 0 & 0 & 0 \\ E_3 & 0 & -E_1 & 0 & 0 \\ E_4 & 0 & 0 & -E_1 & 0 \\ E_5 & 0 & 0 & 0 & -E_1 \end{pmatrix}.$$

In this case, (ϕ, ξ, η, g) is a para-Sasakian structure on M and hence $M(\phi, \xi, \eta, g)$ is a 5-dimensional para-Sasakian manifold.

Using (2.10) and the above equation, one can easily obtain the following:

$$\begin{pmatrix} \tilde{\nabla}_{E_1} E_1 & \tilde{\nabla}_{E_1} E_2 & \tilde{\nabla}_{E_1} E_3 & \tilde{\nabla}_{E_1} E_4 & \tilde{\nabla}_{E_1} E_5 \\ \tilde{\nabla}_{E_2} E_1 & \tilde{\nabla}_{E_2} E_2 & \tilde{\nabla}_{E_2} E_3 & \tilde{\nabla}_{E_2} E_4 & \tilde{\nabla}_{E_2} E_5 \\ \tilde{\nabla}_{E_3} E_1 & \tilde{\nabla}_{E_3} E_2 & \tilde{\nabla}_{E_3} E_3 & \tilde{\nabla}_{E_3} E_4 & \tilde{\nabla}_{E_3} E_5 \\ \tilde{\nabla}_{E_4} E_1 & \tilde{\nabla}_{E_4} E_2 & \tilde{\nabla}_{E_4} E_3 & \tilde{\nabla}_{E_4} E_4 & \tilde{\nabla}_{E_4} E_5 \\ \tilde{\nabla}_{E_5} E_1 & \tilde{\nabla}_{E_5} E_2 & \tilde{\nabla}_{E_5} E_3 & \tilde{\nabla}_{E_5} E_4 & \tilde{\nabla}_{E_5} E_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2E_2 & -2E_1 & 0 & 0 & 0 \\ 2E_3 & 0 & -2E_1 & 0 & 0 \\ 2E_4 & 0 & 0 & -2E_1 & 0 \\ 2E_5 & 0 & 0 & 0 & -2E_1 \end{pmatrix}.$$

From above results it can be easily obtain the non-zero components of Riemannian curvature and Ricci curvature tensors as

$$\begin{aligned} R(E_1, E_2)E_1 &= E_2, & R(E_1, E_2)E_2 &= -E_1, & R(E_1, E_3)E_1 &= E_3, & R(E_1, E_3)E_3 &= -E_1, \\ R(E_1, E_4)E_1 &= E_4, & R(E_1, E_4)E_4 &= -E_1, & R(E_1, E_5)E_1 &= E_5, & R(E_1, E_5)E_5 &= -E_1, \\ R(E_2, E_3)E_2 &= E_3, & R(E_2, E_3)E_3 &= -E_2, & R(E_2, E_4)E_2 &= E_4, & R(E_2, E_4)E_4 &= -E_2, \\ R(E_2, E_5)E_2 &= E_5, & R(E_2, E_5)E_5 &= -E_2, & R(E_3, E_4)E_3 &= E_4, & R(E_3, E_4)E_4 &= -E_3, \\ R(E_3, E_5)E_3 &= E_5, & R(E_3, E_5)E_5 &= -E_3, & R(E_4, E_5)E_4 &= E_5, & R(E_4, E_5)E_5 &= -E_4, \end{aligned}$$

and

$$\begin{aligned} \tilde{R}(E_1, E_2)E_1 &= 2E_2, & \tilde{R}(E_1, E_2)E_2 &= -2E_1, & \tilde{R}(E_1, E_3)E_1 &= 2E_3, & \tilde{R}(E_1, E_3)E_3 &= -2E_1, \\ \tilde{R}(E_1, E_4)E_1 &= 2E_4, & \tilde{R}(E_1, E_4)E_4 &= -2E_1, & \tilde{R}(E_1, E_5)E_1 &= 2E_5, & \tilde{R}(E_1, E_5)E_5 &= -2E_1, \\ \tilde{R}(E_2, E_3)E_2 &= 2E_3, & \tilde{R}(E_2, E_3)E_3 &= -2E_2, & \tilde{R}(E_2, E_4)E_2 &= 2E_4, & \tilde{R}(E_2, E_4)E_4 &= -2E_2, \\ \tilde{R}(E_2, E_5)E_2 &= 2E_5, & \tilde{R}(E_2, E_5)E_5 &= -2E_2, & \tilde{R}(E_3, E_4)E_3 &= 2E_4, & \tilde{R}(E_3, E_4)E_4 &= -2E_3, \\ \tilde{R}(E_3, E_5)E_3 &= 2E_5, & \tilde{R}(E_3, E_5)E_5 &= -2E_3, & \tilde{R}(E_4, E_5)E_4 &= 2E_5, & \tilde{R}(E_4, E_5)E_5 &= -2E_4, & (7.7) \\ \tilde{S}(E_1, E_1) &= \tilde{S}(E_2, E_2) = \tilde{S}(E_3, E_3) = \tilde{S}(E_4, E_4) = \tilde{S}(E_5, E_5) = -8, & & & & & & (7.8) \\ \tilde{S}(X, Y) &= -2(5-1)g(X, Y) = -8g(X, Y), \end{aligned}$$

where $X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4 + f_1E_5$ and $Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4 + f_2E_5$.

From (7.7), (7.8) and the expression of the endomorphism one can easily substantiate, the equation (7.4) and hence second part of the Theorem 2 (for $L_{\tilde{S}} \neq -2$).

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