Bounds for the Generalized (Φ, f) -Mean Difference

SILVESTRU SEVER DRAGOMIR^{1,2}

¹*Mathematics, College of Engineering & Science,
Victoria University, PO Box 14428,
Melbourne City, MC 8001, Australia.*

sever.dragomir@vu.edu.au, <http://rgmia.org/dragomir>

²*School of Computer Science & Applied Mathematics,
University of the Witwatersrand,
Private Bag 3, Johannesburg 2050, South Africa*

ABSTRACT

In this paper we establish some bounds for the (Φ, f) -mean difference introduced in the general settings of measurable spaces and Lebesgue integral, which is a two functions generalization of *Gini mean difference* that has been widely used by economists and sociologists to measure economic inequality.

RESUMEN

En este artículo establecemos algunas cotas para la (Φ, f) -diferencia media introducida en el contexto general de espacios medibles e integral de Lebesgue, que es una generalización a dos funciones de la *diferencia media de Gini* que ha sido ampliamente utilizada por economistas y sociólogos para medir desigualdad económica.

Keywords and Phrases: Gini mean difference, Mean deviation, Lebesgue integral, Expectation, Jensen's integral inequality.

2010 AMS Mathematics Subject Classification: 26D15; 26D10; 94A17.

1. Introduction

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω , a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a ν -measurable function $w : \Omega \rightarrow \mathbb{R}$, with $w(x) \geq 0$ for ν -a.e. (almost every) $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$, consider the *Lebesgue space*

$$L_w(\Omega, \nu) := \{f : \Omega \rightarrow \mathbb{R}, f \text{ is } \nu\text{-measurable and } \int_{\Omega} w(x) |f(x)| d\nu(x) < \infty\}.$$

Let I be an interval of real numbers and $\Phi : I \rightarrow \mathbb{R}$ a Lebesgue measurable function on I . For $f : \Omega \rightarrow I$ a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$ we define the *generalized (Φ, f) -mean difference* $R_G(\Phi, f; w)$ by

$$R_G(\Phi, f; w) := \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| d\nu(x) d\nu(y) \quad (1.1)$$

and the *generalized (Φ, f) -mean deviation* $M_D(\Phi, f; w)$ by

$$M_D(\Phi, f; w) := \int_{\Omega} w(x) |(\Phi \circ f)(x) - E(\Phi, f; w)| d\nu(x), \quad (1.2)$$

where

$$E(\Phi, f; w) := \int_{\Omega} (\Phi \circ f)(y) w(y) d\nu(y)$$

the *generalized (Φ, f) -expectation*.

If $\Phi = e$, where $e(t) = t$, $t \in \mathbb{R}$ is the *identity mapping*, then we can consider the particular cases of interest, the *generalized f -mean difference*

$$R_G(f; w) := R_G(e, f; w) = \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |f(x) - f(y)| d\nu(x) d\nu(y) \quad (1.3)$$

and the *generalized f -mean deviation*

$$M_D(f; w) := M_D(e, f; w) = \int_{\Omega} w(x) |f(x) - E(f; w)| d\nu(x), \quad (1.4)$$

where $E(f; w) := \int_{\Omega} f(y) w(y) d\nu(y)$ is the *generalized f -expectation*.

If $\Omega = [-\infty, \infty]$ and $f = e$ then we have the usual *mean difference*

$$R_G(w) := R_G(f; w) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x) w(y) |x - y| dx dy \quad (1.5)$$

and the *mean deviation*

$$M_D(w) := M_D(f; w) = \int_{\Omega} w(x) |x - E(w)| dx, \quad (1.6)$$

where $w : \mathbb{R} \rightarrow [0, \infty)$ is a *density function*, this means that w is integrable on $\mathbb{R}$ and $\int_{-\infty}^{\infty} w(t) dt = 1$, and

$$E(w) := \int_{-\infty}^{\infty} xw(x) dx \quad (1.7)$$

denote the *expectation* of w provided that the integral exists and is finite.

The mean difference $R_G(w)$ was proposed by Gini in 1912 [21], after whom it is usually named, but was discussed by Helmert and other German writers in the 1870's (cf. H. A. David [13], see also [26, p. 48]). It has a certain theoretical attraction, being dependent on the spread of the variate-values among themselves and not on the deviations from some central value ([26, p. 48]). Further, its defining integral (1.5) may converge when that of the *variance* $\sigma(w)$,

$$\sigma(w) := \int_{-\infty}^{\infty} (x - E(w))^2 w(x) dx, \quad (1.8)$$

does not. It is, however, more difficult to compute than the standard deviation.

For some recent results concerning integral representations and bounds for $R_G(w)$ see [5], [6], [8] and [9].

For instance, if $w : \mathbb{R} \rightarrow [0, \infty)$ is a density function we define by

$$W(x) := \int_{-\infty}^x w(t) dt, \quad x \in \mathbb{R}$$

its *cumulative function*. Then we have [5], [6]:

$$\begin{aligned} R_G(w) &= 2 \operatorname{Cov}(e, W) = \int_{-\infty}^{\infty} (1 - W(y)) W(y) dy \\ &= 2 \int_{-\infty}^{\infty} xw(x) W(x) dx - E(w) \\ &= 2 \int_{-\infty}^{\infty} (x - E(w)) (W(x) - \gamma) w(x) dx \\ &= 2 \int_{-\infty}^{\infty} (x - \delta) \left(W(x) - \frac{1}{2} \right) w(x) dx \end{aligned} \quad (1.9)$$

for any $\gamma, \delta \in \mathbb{R}$ and [6]:

$$R_G(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - y) (W(x) - W(y)) w(x) w(y) dx dy. \quad (1.10)$$

With the above assumptions, we have the bounds [5]:

$$\frac{1}{2} M_D(w) \leq R_G(w) \leq 2 \sup_{x \in \mathbb{R}} |W(x) - \gamma| M_D(w) \leq M_D(w), \quad (1.11)$$

for any $\gamma \in [0, 1]$, where $W(\cdot)$ is the cumulative distribution of w and $M_D(w)$ is the mean deviation.

Consider the n -tuple of real numbers $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$ a probability distribution, i.e. $p_i \geq 0$ for each $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$, then by taking $\Omega = \{1, \dots, n\}$ and the discrete measure, we can consider from (1.1) and (1.2) that (see [7])

$$R_G(\mathbf{a}; \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n p_i p_j |\Phi(a_i) - \Phi(a_j)|, \quad (1.12)$$

and

$$M_D(\mathbf{a}; \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n p_i \left| \Phi(a_i) - \sum_{j=1}^n p_j \Phi(a_j) \right| \quad (1.13)$$

where $\mathbf{a} \in I^n := I \times \dots \times I$ and $\Phi : I \rightarrow \mathbb{R}$.

The quantity $R_G(\mathbf{a}; \mathbf{p})$ has been defined in [7] and some results were obtained.

In the case when $\Phi = e$, then we get the special case of Gini mean difference and mean deviation of an empirical distribution that is particularly important for applications,

$$R_G(\mathbf{a}; \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n p_i p_j |a_i - a_j|, \quad (1.14)$$

and

$$M_D(\mathbf{a}; \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n p_i \left| a_i - \sum_{j=1}^n p_j a_j \right|. \quad (1.15)$$

The following result incorporates an upper bound for the weighted Gini mean difference [7]:

For any $\mathbf{a} \in \mathbb{R}^n$ and any $\mathbf{p}$ a probability distribution, we have the inequality:

$$\frac{1}{2} M_D(\mathbf{a}; \mathbf{p}) \leq R_G(\mathbf{a}; \mathbf{p}) \leq \inf_{\gamma \in \mathbb{R}} \left[\sum_{i=1}^n p_i |a_i - \gamma| \right] \leq M_D(\mathbf{a}; \mathbf{p}). \quad (1.16)$$

The constant $\frac{1}{2}$ in the first inequality in (1.16) is sharp.

For some recent results for discrete Gini mean difference and mean deviation, see [7], [11], [14] and [15].

2. General Bounds

We have:

Theorem 1. *Let I be an interval of real numbers and $\Phi : I \rightarrow \mathbb{R}$ a Lebesgue measurable function on I . If $w : \Omega \rightarrow \mathbb{R}$ is a ν -measurable function with $w(x) \geq 0$ for ν -a.e. (almost every) $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$ and if $f : \Omega \rightarrow I$ is a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then*

$$\frac{1}{2}M_D(\Phi, f; w) \leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w), \quad (2.1)$$

where

$$I(\Phi, f; w) := \inf_{\gamma \in \mathbb{R}} \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| d\nu(x). \quad (2.2)$$

Demostración. Using the properties of the integral, we have

$$\begin{aligned} R_G(\Phi, f; w) &= \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| d\nu(x) d\nu(y) \\ &\geq \frac{1}{2} \int_{\Omega} w(x) \left| (\Phi \circ f)(x) \int_{\Omega} w(y) d\nu(y) - \int_{\Omega} w(y) (\Phi \circ f)(y) d\nu(y) \right| d\nu(x) \\ &= \frac{1}{2} \int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) d\nu(y) \right| d\nu(x) \\ &= \frac{1}{2} M_D(\Phi, f; w) \end{aligned}$$

and the first inequality in (2.1) is proved.

By the triangle inequality for modulus we have

$$\begin{aligned} |(\Phi \circ f)(x) - (\Phi \circ f)(y)| &= |(\Phi \circ f)(x) - \gamma + \gamma - (\Phi \circ f)(y)| \\ &\leq |(\Phi \circ f)(x) - \gamma| + |(\Phi \circ f)(y) - \gamma| \end{aligned} \quad (2.3)$$

for any $x, y \in \Omega$ and $\gamma \in \mathbb{R}$.

Now, if we multiply (2.3) by $\frac{1}{2}w(x)w(y)$ and integrate, we get

$$\begin{aligned}
 R_G(\Phi, f; w) &= \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x) \, d\nu(y) \\
 &\leq \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) [|(\Phi \circ f)(x) - \gamma| + |(\Phi \circ f)(y) - \gamma|] \, d\nu(x) \, d\nu(y) \\
 &= \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) \, d\nu(y) \\
 &\quad + \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(y) - \gamma| \, d\nu(x) \, d\nu(y) \\
 &= \frac{1}{2} \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) + \frac{1}{2} \int_{\Omega} w(y) |(\Phi \circ f)(y) - \gamma| \, d\nu(y) \\
 &= \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) \tag{2.4}
 \end{aligned}$$

for any $\gamma \in \mathbb{R}$.

Taking the infimum over $\gamma \in \mathbb{R}$ in (2.4) we get the second part of (2.1).

Since, obviously

$$\begin{aligned}
 I(\Phi, f; w) &= \inf_{\gamma \in \mathbb{R}} \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) \\
 &\leq \int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right| \, d\nu(x) \\
 &= M_D(\Phi, f; w),
 \end{aligned}$$

the last part of (2.1) is thus proved. ■

By the Cauchy-Bunyakowsky-Schwarz (CBS) inequality, if $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$, then we have

$$\begin{aligned}
 &\left[\int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right| \, d\nu(x) \right]^2 \\
 &\leq \int_{\Omega} w(x) \left[(\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right]^2 \, d\nu(x) \\
 &= \int_{\Omega} w(x) (\Phi \circ f)^2(x) \, d\nu(x) \\
 &\quad - 2 \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \int_{\Omega} w(x) (\Phi \circ f)(x) \, d\nu(x) \\
 &\quad + \left[\int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right]^2 \int_{\Omega} w(x) \, d\nu(x) \\
 &= \int_{\Omega} w(x) (\Phi \circ f)^2(x) \, d\nu(x) - \left[\int_{\Omega} w(x) (\Phi \circ f)(x) \, d\nu(x) \right]^2.
 \end{aligned}$$

By considering the *generalized (Φ, f) -dispersion*

$$\sigma(\Phi, f; w) := \left(\int_{\Omega} w(x) (\Phi \circ f)^2(x) d\nu(x) - \left[\int_{\Omega} w(x) (\Phi \circ f)(x) d\nu(x) \right]^2 \right)^{1/2},$$

then we have

$$M_D(\Phi, f; w) \leq \sigma(\Phi, f; w) \quad (2.5)$$

provided $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$.

If there exists the constants m, M so that

$$-\infty < m \leq \Phi(t) \leq M < \infty \text{ for almost any } t \in I \quad (2.6)$$

then by the reverse CBS inequality

$$\sigma(\Phi, f; w) \leq \frac{1}{2}(M - m), \quad (2.7)$$

by (2.1) and by (2.5) we can state the following result:

Corollary 1. *Let I be an interval of real numbers and $\Phi : I \rightarrow \mathbb{R}$ a Lebesgue measurable function on I satisfying the condition (2.6) for some constants m, M . If $w : \Omega \rightarrow \mathbb{R}$ is a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$ and if $f : \Omega \rightarrow I$ is a ν -measurable function with $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$, then we have the chain of inequalities*

$$\begin{aligned} \frac{1}{2}M_D(\Phi, f; w) &\leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w) \\ &\leq \sigma(\Phi, f; w) \leq \frac{1}{2}(M - m). \end{aligned} \quad (2.8)$$

We observe that, in the discrete case we obtain from (2.1) the inequality (1.16) while for the univariate case with $\int_{-\infty}^{\infty} w(t) dt = 1$ we have

$$\frac{1}{2}M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \quad (2.9)$$

where

$$I(w) := \inf_{\gamma \in \mathbb{R}} \int_{-\infty}^{\infty} w(x) |x - \gamma| dx. \quad (2.10)$$

If w is supported on the finite interval $[a, b]$, namely $\int_a^b w(x) dx = 1$, then we have the chain of inequalities

$$\frac{1}{2}M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \leq \frac{1}{2}(M - m). \quad (2.11)$$

3. Bounds for Various Classes of Functions

In the case of functions of bounded variation we have:

Theorem 2. Let $\Phi : [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation on the closed interval $[a, b]$. If $w : \Omega \rightarrow \mathbb{R}$ is a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$ and if $f : \Omega \rightarrow [a, b]$ is a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then

$$R_G(\Phi, f; w) \leq \frac{1}{2} \bigvee_a^b(\Phi), \quad (3.1)$$

where $\bigvee_a^b(\Phi)$ is the total variation of Φ on $[a, b]$.

Demostración. Using the inequality (2.4) we have

$$R_G(\Phi, f; w) \leq \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| d\nu(x) \quad (3.2)$$

for any $\gamma \in \mathbb{R}$.

By the triangle inequality, we have

$$\begin{aligned} & \left| (\Phi \circ f)(x) - \frac{1}{2} [\Phi(a) + \Phi(b)] \right| \\ & \leq \frac{1}{2} |\Phi(a) - \Phi(f(x))| + \frac{1}{2} |\Phi(b) - \Phi(f(x))| \end{aligned} \quad (3.3)$$

for any $x \in \Omega$.

Since $\Phi : [a, b] \rightarrow \mathbb{R}$ is of bounded variation and d is a division of $[a, b]$, namely

$$d \in \mathcal{D}([a, b]) := \{d := \{a = t_0 < t_1 < \dots < t_n = b\}\},$$

then

$$\bigvee_a^b(\Phi) = \sup_{d \in \mathcal{D}([a, b])} \sum_{i=0}^{n-1} |\Phi(t_{i+1}) - \Phi(t_i)| < \infty.$$

Taking the division $d_0 := \{a = t_0 < t < t_2 = b\}$ we then have

$$|\Phi(t) - \Phi(a)| + |\Phi(b) - \Phi(t)| \leq \bigvee_a^b(\Phi)$$

for any $t \in [a, b]$ and then

$$|\Phi(f(x)) - \Phi(a)| + |\Phi(b) - \Phi(f(x))| \leq \bigvee_a^b(\Phi) \quad (3.4)$$

for any $x \in \Omega$.

On making use of (3.3) and (3.4) we get

$$\left| (\Phi \circ f)(x) - \frac{1}{2} [\Phi(a) + \Phi(b)] \right| \leq \frac{1}{2} \bigvee_a^b(\Phi) \quad (3.5)$$

for any $x \in \Omega$.

If we multiply (3.5) by $w(x)$ and integrate, then we obtain

$$\int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \frac{1}{2} [\Phi(a) + \Phi(b)] \right| dx \leq \frac{1}{2} \bigvee_a^b(\Phi). \quad (3.6)$$

Finally, by choosing $\gamma = \frac{1}{2} [\Phi(a) + \Phi(b)]$ in (3.2) and making use of (3.6) we deduce the desired result (3.1). ■

In the case of absolutely continuous functions we have:

Theorem 3. Let $\Phi : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on the closed interval $[a, b]$. If $w : \Omega \rightarrow \mathbb{R}$ is a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$ and if $f : \Omega \rightarrow [a, b]$ is a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then

$$R_G(\Phi, f; w) \leq \begin{cases} \|\Phi'\|_{[a, b], \infty} R_G(f; w) & \text{if } \Phi' \in L_{\infty}([a, b]), \\ \frac{1}{2^{1/p}} \|\Phi'\|_{[a, b], p} R_G^{1/q}(f; w) & \text{if } \Phi' \in L_p([a, b]), \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \quad (3.7)$$

where the Lebesgue norms are defined by

$$\|g\|_{[\alpha, \beta], p} := \begin{cases} \operatorname{ess\,sup}_{t \in [\alpha, \beta]} |g(t)| & \text{if } p = \infty, \\ \left(\int_{\alpha}^{\beta} |g(t)|^p dt \right)^{1/p} & \text{if } p \geq 1 \end{cases}$$

and $L_p([\alpha, \beta]) := \{g \mid g \text{ measurable and } \|g\|_{[\alpha, \beta], p} < \infty\}$, $p \in [1, \infty]$.

Demostración. Since f is absolutely continuous, then we have

$$\Phi(t) - \Phi(s) = \int_s^t \Phi'(u) du$$

for any $t, s \in [a, b]$.

Using the Hölder integral inequality we have

$$\begin{aligned} |\Phi(t) - \Phi(s)| &= \left| \int_s^t \Phi'(u) du \right| \\ &\leq \begin{cases} \|\Phi'\|_{[a, b], \infty} |t - s| & \text{if } p = \infty, \\ \|\Phi'\|_{[a, b], p} |t - s|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \end{aligned} \quad (3.8)$$

for any $t, s \in [a, b]$.

Using (3.8) we then have

$$\begin{aligned} & |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \\ & \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} |f(x) - f(y)| & \text{if } p = \infty, \\ \|\Phi'\|_{[a,b],p} |f(x) - f(y)|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \end{aligned} \quad (3.9)$$

for any $x, y \in \Omega$.

If we multiply (3.9) by $\frac{1}{2}w(x)w(y)$ and integrate, then we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x) \, d\nu(y) \\ & \leq \begin{cases} \frac{1}{2} \|\Phi'\|_{[a,b],\infty} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) & \text{if } p = \infty, \\ \frac{1}{2} \|\Phi'\|_{[a,b],p} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) & \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \end{aligned} \quad (3.10)$$

This proves the first branch of (3.7).

Using Jensen's integral inequality for concave function $\Psi(t) = t^s$, $s \in (0, 1)$ we have for $s = \frac{1}{q} < 1$ that

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) \\ & \leq \left(\int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q}, \end{aligned}$$

which implies that

$$\begin{aligned} & \frac{1}{2} \|\Phi'\|_{[a,b],p} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) \\ & \leq \frac{1}{2} \|\Phi'\|_{[a,b],p} \left(\int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\ & = \|\Phi'\|_{[a,b],p} \left(\frac{1}{2^q} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\ & = \|\Phi'\|_{[a,b],p} \left(\frac{1}{2^{q-1}} \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\ & = \frac{1}{2^{\frac{q-1}{q}}} \|\Phi'\|_{[a,b],p} (R_G(f; w))^{1/q} = \frac{1}{2^{1/p}} \|\Phi'\|_{[a,b],p} R_G^{1/q}(f; w) \end{aligned}$$

and the second part of (3.7) is proved. ■

The function $\Phi : [a, b] \rightarrow \mathbb{R}$ is called of r -H-Hölder type with the given constants $r \in (0, 1]$ and $H > 0$ if

$$|\Phi(t) - \Phi(s)| \leq H|t - s|^r$$

for any $t, s \in [a, b]$.

In the case when $r = 1$, namely, there is the constant $L > 0$ such that

$$|\Phi(t) - \Phi(s)| \leq L|t - s|$$

for any $t, s \in [a, b]$, the function Φ is called L -Lipschitzian on $[a, b]$.

We have:

Theorem 4. Let $\Phi : [a, b] \rightarrow \mathbb{R}$ be a function of r -H-Hölder type on the closed interval $[a, b]$. If $w : \Omega \rightarrow \mathbb{R}$ is a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$ and if $f : \Omega \rightarrow [a, b]$ is a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then

$$R_G(\Phi, f; w) \leq \frac{1}{2^{1-r}} HR_G^r(f; w). \quad (3.11)$$

In particular, if Φ is L -Lipschitzian on $[a, b]$, then

$$R_G(\Phi, f; w) \leq LR_G(f; w). \quad (3.12)$$

Demostración. We have

$$|(\Phi \circ f)(x) - (\Phi \circ f)(y)| \leq H|f(x) - f(y)|^r \quad (3.13)$$

for any $x, y \in \Omega$.

If we multiply (3.13) by $\frac{1}{2}w(x)w(y)$ and integrate, then we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| d\nu(x) d\nu(y) \\ & \leq \frac{1}{2} H \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^r d\nu(x) d\nu(y). \end{aligned} \quad (3.14)$$

By Jensen's integral inequality for concave functions we also have

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)|^r d\nu(x) d\nu(y) \\ & \leq \left(\int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| d\nu(x) d\nu(y) \right)^r. \end{aligned} \quad (3.15)$$

Therefore, by (3.14) and (3.15) we get

$$\begin{aligned} R_G(\Phi, f; w) & \leq \frac{1}{2} H \left(\int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| d\nu(x) d\nu(y) \right)^r \\ & = \frac{1}{2^{1-r}} H \left(\frac{1}{2} \int_{\Omega} \int_{\Omega} w(x)w(y) |f(x) - f(y)| d\nu(x) d\nu(y) \right)^r \\ & = \frac{1}{2^{1-r}} HR_G^r(f; w) \end{aligned}$$

and the inequality (3.11) is proved. ■

We have:

Theorem 5. Let $\Phi, \Psi : [a, b] \rightarrow \mathbb{R}$ be continuous functions on $[a, b]$ and differentiable on (a, b) with $\Psi'(t) \neq 0$ for $t \in (a, b)$. If $w : \Omega \rightarrow \mathbb{R}$ is a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$ and if $f : \Omega \rightarrow [a, b]$ is a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then

$$\inf_{t \in (a, b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| R_G(\Psi, f; w) \leq R_G(\Phi, f; w) \leq \sup_{t \in (a, b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| R_G(\Psi, f; w). \quad (3.16)$$

Demostración. By the Cauchy's mean value theorem, for any $t, s \in [a, b]$ with $t \neq s$ there exists a ξ between t and s such that

$$\frac{\Phi(t) - \Phi(s)}{\Psi(t) - \Psi(s)} = \frac{\Phi'(\xi)}{\Psi'(\xi)}.$$

This implies that

$$\begin{aligned} \inf_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(t) - \Psi(s)| &\leq |\Phi(t) - \Phi(s)| \\ &\leq \sup_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(t) - \Psi(s)| \end{aligned} \quad (3.17)$$

for any $t, s \in [a, b]$.

Therefore, we have

$$\begin{aligned} \inf_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(f(x)) - \Psi(f(y))| &\leq |\Phi(f(x)) - \Phi(f(y))| \\ &\leq \sup_{t \in (a, b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| |\Psi(f(x)) - \Psi(f(y))| \end{aligned} \quad (3.18)$$

for any $x, y \in \Omega$.

If we multiply (3.18) by $\frac{1}{2}w(x)w(y)$ and integrate, we get the desired result (3.16). ■

Corollary 2. Let $\Phi : [a, b] \rightarrow \mathbb{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b) . If w is as in Theorem 5, then we have

$$\inf_{t \in (a, b)} |\Phi'(t)| R_G(f; w) \leq R_G(\Phi, f; w) \leq \sup_{t \in (a, b)} |\Phi'(t)| R_G(f; w). \quad (3.19)$$

We also have:

Theorem 6. Let $\Phi : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on the closed interval $[a, b]$. If $w : \Omega \rightarrow \mathbb{R}$ is a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$

and if $f: \Omega \rightarrow [a, b]$ is a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then

$$\begin{aligned} R_G(\Phi, f; w) &\leq \begin{cases} \|\Phi'\|_{[a, b], \infty} M(f; w) & \text{if } p = \infty, \\ \|\Phi'\|_{[a, b], p} M^{1/q}(f; w) & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \\ &\leq \begin{cases} \frac{1}{2}(b-a) \|\Phi'\|_{[a, b], \infty} & \text{if } p = \infty, \\ \frac{1}{2^{1/q}}(b-a)^{1/q} \|\Phi'\|_{[a, b], p} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \end{aligned} \quad (3.20)$$

where $M(f; w)$ is defined by

$$M(f; w) := \int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x). \quad (3.21)$$

Demostración. From the inequality (3.8) we have

$$\begin{aligned} &|(\Phi \circ f)(x) - \Phi\left(\frac{a+b}{2}\right)| \\ &\leq \begin{cases} \|\Phi'\|_{[a, b], \infty} \left| f(x) - \frac{a+b}{2} \right| & \text{if } p = \infty, \\ \|\Phi'\|_{[a, b], p} \left| f(x) - \frac{a+b}{2} \right|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \end{aligned} \quad (3.22)$$

for any $x \in \Omega$.

Now, if we multiply (3.22) by $w(x)$ and integrate, then we get

$$\begin{aligned} &\int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \Phi\left(\frac{a+b}{2}\right) \right| d\nu(x) \\ &\leq \begin{cases} \|\Phi'\|_{[a, b], \infty} \int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x) & \text{if } p = \infty, \\ \|\Phi'\|_{[a, b], p} \int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right|^{1/q} d\nu(x) & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \end{aligned} \quad (3.23)$$

By Jensen's integral inequality for concave functions we have

$$\int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right|^{1/q} d\nu(x) \leq \left(\int_{\Omega} w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x) \right)^{1/q}. \quad (3.24)$$

On making use of (3.2), (3.23) and (3.24) we get the first inequality in (3.20).

The last part of (3.20) follows by the fact that

$$\left| f(x) - \frac{a+b}{2} \right| \leq \frac{1}{2}(b-a)$$

for any $x \in \Omega$. ■

4. Bounds for Special Convexity

When some convexity properties for the function Φ are assumed, then other bounds can be derived as follows.

Theorem 7. Let $w : \Omega \rightarrow \mathbb{R}$ be a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$ and $f : \Omega \rightarrow [a, b]$ be a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$. Assume also that $\Phi : [a, b] \rightarrow \mathbb{R}$ is a continuous function on $[a, b]$.

(i) If $|\Phi|$ is concave on $[a, b]$, then

$$R_G(\Phi, f; w) \leq |\Phi(E(f; w))|, \quad (4.1)$$

(ii) If $|\Phi|$ is convex on $[a, b]$, then

$$R_G(\Phi, f; w) \leq \frac{1}{b-a} [(b - E(f; w)) |\Phi(a)| + (E(f; w) - a) \Phi(b)]. \quad (4.2)$$

Demostración. (i) If $|\Phi|$ is concave on $[a, b]$, then by Jensen's inequality we have

$$\int_{\Omega} w(x) |(\Phi \circ f)(x)| d\nu(x) \leq \left| \Phi \left(\int_{\Omega} w(x) f(x) d\nu(x) \right) \right|. \quad (4.3)$$

From (3.2) for $\gamma = 0$ we also have

$$R_G(\Phi, f; w) \leq \int_{\Omega} w(x) |(\Phi \circ f)(x)| d\nu(x). \quad (4.4)$$

This is an inequality of interest in itself.

On utilizing (4.3) and (4.4) we get (4.1).

(ii) Since $|\Phi|$ is convex on $[a, b]$, then for any $t \in [a, b]$ we have

$$|\Phi(t)| = \left| \Phi \left(\frac{(b-t)a + b(t-a)}{b-a} \right) \right| \leq \frac{(b-t)|\Phi(a)| + (t-a)\Phi(b)}{b-a}.$$

This implies that

$$|(\Phi \circ f)(x)| \leq \frac{(b-f(x))|\Phi(a)| + (f(x)-a)\Phi(b)}{b-a} \quad (4.5)$$

for any $x \in \Omega$.

If we multiply (4.5) by $w(x)$ and integrate, then we get

$$\begin{aligned} & \int_{\Omega} w(x) |(\Phi \circ f)(x)| d\nu(x) \\ & \leq \frac{1}{b-a} \left[\left(b \int_{\Omega} w(x) d\nu(x) - \int_{\Omega} w(x) f(x) d\nu(x) \right) |\Phi(a)| \right. \\ & \quad \left. + \left(\int_{\Omega} w(x) f(x) d\nu(x) - a \int_{\Omega} w(x) d\nu(x) \right) \Phi(b) \right], \end{aligned}$$

which, together with (4.4), produces the desired result (4.2). ■

In order to state other results we need the following definitions:

Definition 1 ([19]). We say that a function $f: I \rightarrow \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

It is important to note that $P(I)$ contains all nonnegative monotone, convex and *quasi convex functions*, i.e. functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P -functions see [19] and [28] while for quasi convex functions, the reader can consult [18].

Definition 2 ([3]). Let s be a real number, $s \in (0, 1]$. A function $f: [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex (in the second sense) or Breckner s -convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [3], [4], [16], [17], [25], [27] and [29].

Theorem 8. Let $w: \Omega \rightarrow \mathbb{R}$ be a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$ and $f: \Omega \rightarrow [a, b]$ be a ν -measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$. Assume also that $\Phi: [a, b] \rightarrow \mathbb{R}$ is a continuous function on $[a, b]$.

(i) If $|\Phi|$ belongs to the class P on $[a, b]$, then

$$R_G(\Phi, f; w) \leq |\Phi(a)| + \Phi(b); \quad (4.6)$$

(ii) If $|\Phi|$ is quasi convex on $[a, b]$, then

$$R_G(\Phi, f; w) \leq \max\{|\Phi(a)|, \Phi(b)\}; \quad (4.7)$$

(iii) If $|\Phi|$ is Breckner s -convex on $[a, b]$, then

$$\begin{aligned} R_G(\Phi, f; w) &\leq \frac{1}{(b-a)^s} \left[|\Phi(a)| \int_{\Omega} w(x) (b-f(x))^s d\nu(x) \right. \\ &\quad \left. + \Phi(b) \int_{\Omega} w(x) (f(x)-a)^s d\nu(x) \right] \\ &\leq \frac{1}{(b-a)^s} \left[|\Phi(a)| (b-E(f; w))^s d\nu(x) \right. \\ &\quad \left. + \Phi(b) (E(f; w)-a)^s d\nu(x) \right]. \end{aligned} \quad (4.8)$$

Demostración. (i) Since $|\Phi|$ belongs to the class P on $[a, b]$, then for any $t \in [a, b]$ we have

$$|\Phi(t)| = \left| \Phi \left(\frac{(b-t)a + b(t-a)}{b-a} \right) \right| \leq |\Phi(a)| + |\Phi(b)|.$$

This implies that

$$|(\Phi \circ f)(x)| \leq |\Phi(a)| + |\Phi(b)| \quad (4.9)$$

for any $x \in \Omega$.

If we multiply (4.9) by $w(x)$ and integrate, then we get

$$\int_{\Omega} w(x) |(\Phi \circ f)(x)| \, dv(x) \leq |\Phi(a)| + |\Phi(b)|, \quad (4.10)$$

which, together with (4.4), produces the desired result (4.6).

(ii) Goes in a similar way.

(iii) By Breckner s -convexity we have

$$|\Phi(t)| = \left| \Phi \left(\frac{(b-t)a + b(t-a)}{b-a} \right) \right| \leq \left(\frac{b-t}{b-a} \right)^s |\Phi(a)| + \left(\frac{t-a}{b-a} \right)^s |\Phi(b)|$$

for any $t \in [a, b]$.

This implies that

$$|(\Phi \circ f)(x)| \leq \frac{1}{(b-a)^s} [(b-f(x))^s |\Phi(a)| + (f(x)-a)^s |\Phi(b)|] \quad (4.11)$$

for any $x \in \Omega$.

If we multiply (4.11) by $w(x)$ and integrate, then we get

$$\begin{aligned} \int_{\Omega} w(x) |(\Phi \circ f)(x)| \, dv(x) &\leq \frac{1}{(b-a)^s} \left[|\Phi(a)| \int_{\Omega} w(x) (b-f(x))^s \, dv(x) \right. \\ &\quad \left. + |\Phi(b)| \int_{\Omega} w(x) (f(x)-a)^s \, dv(x) \right], \end{aligned} \quad (4.12)$$

which, together with (4.4), produces the first part of (4.8).

The last part follows by Jensen's integral inequality for concave functions, namely

$$\int_{\Omega} w(x) (b-f(x))^s \, dv(x) \leq \left(b - \int_{\Omega} w(x) f(x) \, dv(x) \right)^s$$

and

$$\int_{\Omega} w(x) (f(x)-a)^s \, dv(x) \leq \left(\int_{\Omega} w(x) f(x) \, dv(x) - a \right)^s,$$

where $s \in (0, 1)$. ■

5. Some Examples

Let $f : \Omega \rightarrow [0, \infty)$ be a ν -measurable function and $w : \Omega \rightarrow \mathbb{R}$ a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$. We define, for the function $\Phi(t) = t^p$, $p > 0$, the *generalized (p, f) -mean difference* $R_G(p, f; w)$ by

$$R_G(p, f; w) := \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |f^p(x) - f^p(y)| d\nu(x) d\nu(y) \quad (5.1)$$

and the *generalized (p, f) -mean deviation* $M_D(p, f; w)$ by

$$M_D(p, f; w) := \int_{\Omega} w(x) |f^p(x) - E(p, f; w)| d\nu(x), \quad (5.2)$$

where

$$E(p, f; w) := \int_{\Omega} f^p(y) w(y) d\nu(y) \quad (5.3)$$

is the *generalized (p, f) -expectation*.

If $f : \Omega \rightarrow [a, b] \subset [0, \infty)$ is a ν -measurable function, then by (3.1) we have

$$R_G(p, f; w) \leq \frac{1}{2} (b^p - a^p). \quad (5.4)$$

By (3.7) we have

$$R_G(p, f; w) \leq p \delta_p(a, b) R_G(f; w), \quad (5.5)$$

where

$$\delta_p(a, b) := \begin{cases} b^{p-1} & \text{if } p \geq 1, \\ a^{p-1} & \text{if } p \in (0, 1) \end{cases}$$

and

$$R_G(p, f; w) \leq \frac{p}{2^{1/\alpha}} \left[\frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right]^{1/\alpha} R_G^{1/\beta}(f; w), \quad (5.6)$$

where $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

From (3.20) we also have

$$\begin{aligned} & R_G(p, f; w) \\ & \leq \begin{cases} \delta_p(a, b) M(f; w), \\ p \left(\frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right)^{1/\alpha} M^{1/\beta}(f; w) & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \end{cases} \\ & \leq \begin{cases} \frac{1}{2} (b - a) \delta_p(a, b), \\ \frac{1}{2^{1/\beta}} (b - a)^{1/\beta} p \left(\frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right)^{1/\alpha} & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \end{cases} \end{aligned} \quad (5.7)$$

where $M(f; w)$ is defined by (3.21).

If $p \in (0, 1)$, then the function $|\Phi(t)| = t^p$ is concave on $[a, b] \subset [0, \infty)$ and by (4.1) we have

$$R_G(p, f; w) \leq E^p(f; w). \quad (5.8)$$

For $p \geq 1$ the function $|\Phi(t)| = t^p$ is convex on $[a, b] \subset [0, \infty)$ and by (4.2) we have

$$R_G(p, f; w) \leq \frac{1}{b-a} [(b - E(f; w)) a^p + (E(f; w) - a) b^p]. \quad (5.9)$$

Let $f: \Omega \rightarrow [0, \infty)$ be a ν -measurable function and $w: \Omega \rightarrow \mathbb{R}$ a ν -measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x) d\nu(x) = 1$. We define, for the function $\Phi(t) = \ln t$, the *generalized $(\ln, f)$ -mean difference* $R_G(\ln, f; w)$ by

$$R_G(\ln, f; w) := \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |\ln f(x) - \ln f(y)| d\nu(x) d\nu(y) \quad (5.10)$$

and the *generalized (p, f) -mean deviation* $M_D(\ln, f; w)$ by

$$M_D(\ln, f; w) := \int_{\Omega} w(x) |\ln f(x) - E(\ln, f; w)| d\nu(x), \quad (5.11)$$

where

$$E(\ln, f; w) := \int_{\Omega} w(y) \ln f(y) d\nu(y) \quad (5.12)$$

is the *generalized $(\ln, f)$ -expectation*.

If $f: \Omega \rightarrow [a, b] \subset [0, \infty)$ is a ν -measurable function, then by (3.1) we have

$$R_G(\ln, f; w) \leq \frac{1}{2} (\ln b - \ln a). \quad (5.13)$$

By (3.7) we have

$$R_G(\ln, f; w) \leq \begin{cases} \frac{1}{a} R_G(f; w), \\ \frac{1}{2^{1/p}} \left(\frac{b^{p-1} - a^{p-1}}{(p-1)b^{p-1}a^{p-1}} \right)^{1/p} R_G^{1/q}(f; w) \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \quad (5.14)$$

By (3.20) we have

$$R_G(\ln, f; w) \leq \begin{cases} \frac{1}{a} M(f; w), \\ \left(\frac{b^{p-1} - a^{p-1}}{(p-1)b^{p-1}a^{p-1}} \right)^{1/p} M^{1/q}(f; w) \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \quad (5.15)$$

$$\leq \begin{cases} \frac{1}{2} \left(\frac{b}{a} - 1 \right), \\ \frac{1}{2^{1/q}} (b - a)^{1/q} \left(\frac{b^{p-1} - a^{p-1}}{(p-1)b^{p-1}a^{p-1}} \right)^{1/p} \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

Now, observe that the function $|\Phi(t)| = |\ln t|$ is convex on $(0, 1)$ and concave on $[1, \infty)$. If $f : \Omega \rightarrow [a, b] \subset (0, 1)$ is a ν -measurable function, then by (4.2) we have

$$R_G(\ln, f; w) \leq \frac{1}{b-a} [(b - E(f; w)) |\ln a| + (E(f; w) - a) |\ln b|] \quad (5.16)$$

and if $f : \Omega \rightarrow [a, b] \subset [1, \infty)$, then by (4.1) we have

$$R_G(\ln, f; w) \leq \ln(E(f; w)). \quad (5.17)$$

The interested reader may state similar bounds for functions Φ such as $\Phi(t) = \exp t$, $t \in \mathbb{R}$ or $\Phi(t) = t \ln t$, $t > 0$. We omit the details.

Acknowledgement. The author would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the paper.

Referencias

- [1] M. Alomari and M. Darus, *The Hadamard's inequality for s -convex function*. Int. J. Math. Anal. (Ruse) **2** (2008), no. 13-16, 639–646.
- [2] M. Alomari and M. Darus, *Hadamard-type inequalities for s -convex functions*. Int. Math. Forum **3** (2008), no. 37-40, 1965–1975.
- [3] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*. (German) Publ. Inst. Math. (Beograd) (N.S.) **23(37)** (1978), 13–20.
- [4] W. W. Breckner and G. Orbán, *Continuity Properties of Rationally s -Convex Mappings with v Values in an Ordered Topological Linear Space*. Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.
- [5] P. Cerone and S. S. Dragomir, Bounds for the Gini mean difference via the Sonin identity, *Comp. Math. Modelling*, **50** (2005), 599-609.
- [6] P. Cerone and S. S. Dragomir, Bounds for the Gini mean difference via the Korkine identity, *J. Appl. Math. & Computing*, **22** (2006), no. 3, 305–315.
- [7] P. Cerone and S. S. Dragomir, Bounds for the Gini mean difference of an empirical distribution. *Appl. Math. Lett.* **19** (2006), no. 3, 283–293.
- [8] P. Cerone and S. S. Dragomir, Bounds for the Gini mean difference of continuous distributions defined on finite intervals (I), *Appl. Math. Lett.* **20** (2007), no. 7, 782–789.
- [9] P. Cerone and S. S. Dragomir, Bounds for the Gini mean difference of continuous distributions defined on finite intervals (II), *Comput. Math. Appl.* **52** (2006), no. 10-11, 1555–1562.
- [10] P. Cerone and S. S. Dragomir, A survey on bounds for the Gini mean difference. *Advances in inequalities from probability theory and statistics*, 81–111, Adv. Math. Inequal. Ser., Nova Sci. Publ., New York, 2008.
- [11] P. Cerone and S. S. Dragomir, Bounds for the r -weighted Gini mean difference of an empirical distribution. *Math. Comput. Modelling* **49** (2009), no. 1-2, 180–188.
- [12] X.-L. Cheng and J. Sun, A note on the perturbed trapezoid inequality, *J. Inequal. Pure & Appl. Math.*, 3(2) (2002), Article. 29.
- [13] H. A. David, Gini's mean difference rediscovered, *Biometrika*, **55** (1968), 573.
- [14] S. S. Dragomir, Weighted f -Gini mean difference for convex and symmetric functions in linear spaces. *Comput. Math. Appl.* 60 (2010), no. 3, 734–743.

- [15] S. S. Dragomir, Bounds in terms of Gâteaux derivatives for the weighted f -Gini mean difference in linear spaces. *Bull. Aust. Math. Soc.* **83** (2011), no. 3, 420–434.
- [16] S. S. Dragomir and S. Fitzpatrick, *The Hadamard inequalities for s -convex functions in the second sense*. *Demonstratio Math.* **32** (1999), no. 4, 687–696.
- [17] S. S. Dragomir and S. Fitzpatrick, *The Jensen inequality for s -Breckner convex functions in linear spaces*. *Demonstratio Math.* **33** (2000), no. 1, 43–49.
- [18] S. S. Dragomir and C. E. M. Pearce, *Quasi-convex functions and Hadamard's inequality*, *Bull. Austral. Math. Soc.* **57** (1998), 377–385.
- [19] S. S. Dragomir, J. Pečarić and L. Persson, *Some inequalities of Hadamard type*. *Soochow J. Math.* **21** (1995), no. 3, 335–341.
- [20] J. L. Gastwirth, The estimation of the Lorentz curve and Gini index, *Rev. Econom. Statist.*, **54** (1972), 305–316.
- [21] C. Gini, Variabilità e Metabilità, contributo allo studia della distribuzioni e relationi statistiche, *Studi Economica-Gicenitrici dell' Univ. di Coglani*, **3** (1912), art 2, 1–158.
- [22] G. M. Giorgi, Bibliographic portrait of the Gini concentration ratio, *Metron*, **XLVIII**(1-4) (1990), 103–221.
- [23] G. M. Giorgi, Alcune considerazioni teoriche su di un vecchio ma per sempre attuale indice: il rapporto di concentrazione del Gini, *Metron*, **XLII**(3-4) (1984), 25–40.
- [24] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press.
- [25] H. Hudzik and L. Maligranda, *Some remarks on s -convex functions*. *Aequationes Math.* **48** (1994), no. 1, 100–111.
- [26] M. Kendal and A. Stuart *The Advanced Theory of Statistics*, Volume 1, Distribution Theory, Fourth Edition, Charles Griffin & Comp. Ltd., London, 1977.
- [27] U. S. Kirmaci, M. Klaričić Bakula, M. E. Özdemir and J. Pečarić, *Hadamard-type inequalities for s -convex functions*. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.
- [28] C. E. M. Pearce and A. M. Rubinov, *P -functions, quasi-convex functions, and Hadamard-type inequalities*. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.
- [29] E. Set, M. E. Özdemir and M. Z. Sarikaya, *New inequalities of Ostrowski's type for s -convex functions in the second sense with applications*. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.