Generalized trace pseudo-spectrum of matrix pencils

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ABSTRACT

The objective of the study was to investigate a new notion of generalized trace pseudo-spectrum for an matrix pencils. In particular, we prove many new interesting properties of the generalized trace pseudo-spectrum. In addition, we show an analogue of the spectral mapping theorem for the generalized trace pseudo-spectrum in the matrix algebra.

RESUMEN

El objetivo de este estudio es investigar una nueva noción de pseudo-espectro traza generalizado para pinceles de matrices. En particular, demostramos variadas propiedades nuevas e interesantes del pseudo-espectro traza generalizado. Adicionalmente, mostramos un análogo del teorema espectral de aplicaciones para el pseudo-espectro traza generalizado en el álgebra de matrices.

Keywords and Phrases: pseudo-spectrum, condition pseudo-spectrum, trace pseudo-spectrum.

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1 Introduction

Let $\mathcal{M}_n(\mathbb{C})$ ($\mathcal{M}_n(\mathbb{R})$) denote the algebra of all $n \times n$ complex (real) matrices, $\mathcal{I}$ denotes the $n \times n$ identity matrix and the conjugate transpose of $\mathcal{U}$ is denoted by $\mathcal{U}^*$. We denote by Tr , (resp. Det) the trace (resp. determinant) map on $\mathcal{M}_n(\mathbb{C})$. In the present paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$\mathcal{U}\mathbf{x} = \lambda\mathcal{V}\mathbf{x}.$$

Next, let $\lambda \in \mathbb{C}$ and

$$s_n(\lambda\mathcal{V} - \mathcal{U}) \leq \dots \leq s_2(\lambda\mathcal{V} - \mathcal{U}) \leq s_1(\lambda\mathcal{V} - \mathcal{U})$$

be the singular values of the matrix pencils $\lambda\mathcal{V} - \mathcal{U}$ where $s_1(\lambda\mathcal{V} - \mathcal{U})$ is the smallest and $s_n(\lambda\mathcal{V} - \mathcal{U})$ is largest singular values of the matrix pencil. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, then the set of all eigenvalues of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is denoted by $\sigma(\mathcal{U}, \mathcal{V})$ and is defined as

$$\sigma(\mathcal{U}, \mathcal{V}) = \left\{ \lambda \in \mathbb{C} : \lambda\mathcal{V} - \mathcal{U} \text{ is not invertible} \right\},$$

and its spectral radius by

$$r(\mathcal{U}, \mathcal{V}) = \sup \left\{ |\lambda| : \lambda \in \sigma(\mathcal{U}, \mathcal{V}) \right\}.$$

For an $n \times n$ complex matrices $\mathcal{U}$ and $\mathcal{V}$ and a non-negative real number ε , the pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is defined as the following closed set in the complex plane

$$\sigma_\varepsilon(\mathcal{U}, \mathcal{V}) = \left\{ \lambda \in \mathbb{C} : s_n(\lambda\mathcal{V} - \mathcal{U}) \leq \varepsilon \right\}.$$

Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $0 < \varepsilon < 1$. The condition pseudo-spectrum of the matrix pencils $\lambda\mathcal{V} - \mathcal{U}$ is denoted by $\Sigma_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$\Sigma_\varepsilon(\mathcal{U}, \mathcal{V}) = \left\{ \lambda \in \mathbb{C} : s_n(\lambda\mathcal{V} - \mathcal{U}) \leq \varepsilon s_1(\lambda\mathcal{V} - \mathcal{U}) \right\}.$$

Let ε be a small positive number. For an operator $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, recall that the determinant spectrum of matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is the set $d_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$d_\varepsilon(\mathcal{U}, \mathcal{V}) = \left\{ \lambda \in \mathbb{C} : |\det(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon \right\}.$$

The analysis of eigenvalues and eigenvectors has had a great effect on mathematics, science, engineering, and many other fields. Then, there are countless applications for this type of analysis. The study of matrix pencils is by now a very thoughtful subject, with the notion of pseudospectrum playing a key role in the theory. However, matrix pencils play an important role in numerical linear algebra, perturbation theory, generalized eigenvalue problems. In this paper, we interest by a generalization of eigenvalues called generalized trace pseudo-spectrum for an element in the matrix

algebra to give more information about the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$. For more information on various details on the above concepts, properties and applications of pseudo-spectrum [2, 3, 6, 7, 9], condition spectrum [1, 4, 5] and determinant spectrum [8]. Now, we introduce the new concept of the generalized trace pseudo-spectrum in the following definition.

Definition 1.1. For $\varepsilon > 0$, the generalized trace pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U} \in \mathcal{M}_n(\mathbb{C})$ is denoted by $\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \sigma(\mathcal{U}, \mathcal{V}) \bigcup \left\{ \lambda \in \mathbb{C} : |\text{Tr}(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon \right\}.$$

The generalized trace pseudoresolvent of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is denoted by $\text{Tr}_\rho(\mathcal{U}, \mathcal{V})$ and is defined as

$$\text{Tr}_\rho(\mathcal{U}, \mathcal{V}) = \rho(\mathcal{U}, \mathcal{V}) \bigcap \left\{ \lambda \in \mathbb{C} : |\text{Tr}(\lambda\mathcal{V} - \mathcal{U})| > \varepsilon \right\}.$$

The singular values of a the matrix pencil are important not only for their role in diagonalization but also for their utility in a variety of applications. Since $\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ use all the singular values of $\lambda\mathcal{V}-\mathcal{U}$ to get defined, it is expected to give more information about $\mathcal{U}, \mathcal{V}$ than pseudo-spectrum and condition spectrum. Since the definition use idea of "Trace" the generalization of eigenvalues defined above is named as generalized trace pseudo-spectrum. It is easily seen that the map

$$\mathcal{U} \rightarrow \text{Tr}(\mathcal{U})$$

is continuous linear functional. Here, some important properties of the trace of $\mathcal{U}, \mathcal{B} \in \mathcal{M}_n(\mathbb{C})$ are

$$\text{Tr}(\mathcal{U}\mathcal{B}) = \text{Tr}(\mathcal{B}\mathcal{U}),$$

$$\text{Tr}(\alpha\mathcal{U}) = \alpha\text{Tr}(\mathcal{U}) \quad \text{with } \alpha \in \mathbb{C},$$

$$\text{Tr}(\mathcal{U} + \mathcal{B}) = \text{Tr}(\mathcal{U}) + \text{Tr}(\mathcal{B}).$$

An outline of this paper is the following. In Section 2, we focuses on a new description of the generalized trace pseudo-spectra. Not only do we give a characterization of the generalized trace pseudo-spectrum in the matrix algebra. but also we investigate the connection between generalized trace pseudo-spectrum and algebraic multiplicity of the eigenvalues. In Section 3, we give an analogue of the spectral mapping theorem for the generalized trace pseudo-spectrum in the matrix algebra.

2 Generalized trace pseudo-spectrum.

In this section, some relevant properties of the generalized trace pseudo-spectrum are discussed in detail. For $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, the generalized trace pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is denoted by $\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \sigma(\mathcal{U}, \mathcal{V}) \bigcup \left\{ \lambda \in \mathbb{C} : |\text{Tr}(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon \right\}.$$

The generalized trace pseudo-resolvent of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is denoted by $\text{Trp}_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$\text{Trp}_\varepsilon(\mathcal{U}, \mathcal{V}) = \rho(\mathcal{U}, \mathcal{V}) \cap \{\lambda \in \mathbb{C} : |\text{Tr}(\lambda\mathcal{V} - \mathcal{U})| > \varepsilon\}$$

while the generalized trace pseudo-spectral radius of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is defined as

$$\text{Trr}_\varepsilon(\mathcal{U}, \mathcal{V}) := \sup \{|\lambda| : \lambda \in \text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})\}.$$

Remark 2.1. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$. Then, if $\mathcal{V}$ is nonsingular, then it is possible to reduce the generalized trace pseudo-spectrum to a standard trace pseudo-spectrum for the matrices $\mathcal{V}^{-1}\mathcal{U}$ or $\mathcal{U}\mathcal{V}^{-1}$. i.e.

$$\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \sigma(\mathcal{V}^{-1}\mathcal{U}, \mathcal{I}) \cup \{\lambda \in \mathbb{C} : |\text{Tr}(\lambda - \mathcal{V}^{-1}\mathcal{U})| \leq \varepsilon\},$$

or

$$\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \sigma(\mathcal{U}\mathcal{V}^{-1}, \mathcal{I}) \cup \{\lambda \in \mathbb{C} : |\text{Tr}(\lambda - \mathcal{U}\mathcal{V}^{-1})| \leq \varepsilon\}.$$

The following theorem gives some properties of the generalized trace pseudo-spectrum that follow in a straightforward manner from the definition of the generalized trace pseudo-spectrum.

Theorem 2.1. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,

- (i) $\text{Tr}_0(\mathcal{U}, \mathcal{V}) = \bigcap_{\varepsilon > 0} \text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$.
- (ii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\text{Tr}_{\varepsilon_1}(\mathcal{U}, \mathcal{V}) \subset \text{Tr}_{\varepsilon_2}(\mathcal{U}, \mathcal{V})$.
- (iii) $\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ is a non-empty compact subset of $\mathbb{C}$.
- (iv) If $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus \{0\}$, then $\text{Tr}_\varepsilon(\beta\mathcal{U} + \alpha\mathcal{V}, \mathcal{V}) = \beta \text{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U}, \mathcal{V}) + \alpha$.
- (v) $\text{Tr}_\varepsilon(\alpha\mathcal{V}, \mathcal{V}) = \left\{ \lambda \in \mathbb{C} : |\lambda - \alpha| \leq \frac{\varepsilon}{|\text{Tr}(\mathcal{V})|} \right\}$ for all $\lambda, \alpha \in \mathbb{C}$.

Proof. The proofs of items (i) and (ii) are clear from the definition of generalized trace pseudo-spectrum.

(iii) Using the continuity from $\mathbb{C}$ to $[0, \infty[$ of the map

$$\lambda \rightarrow |\text{Tr}(\lambda\mathcal{V} - \mathcal{U})|,$$

we get that $\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ is a compact set in the complex plane containing the eigenvalues of the matrix pencils $\lambda\mathcal{V} - \mathcal{U}$.

(iv) In fact, it is well know

$$\begin{aligned}\mathrm{Tr}_\varepsilon(\beta\mathcal{U} + \alpha\mathcal{V}, \mathcal{V}) &= \left\{ \lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V} - \beta\mathcal{U} - \alpha\mathcal{V})| \leq \varepsilon \right\} \\ &= \left\{ \lambda \in \mathbb{C} : |\beta| \left| \mathrm{Tr} \left(\frac{\lambda - \alpha}{\beta} \mathcal{V} - \mathcal{U} \right) \right| \leq \varepsilon \right\} \\ &= \left\{ \lambda \in \mathbb{C} : \left| \mathrm{Tr} \left(\frac{\lambda - \alpha}{\beta} \mathcal{V} - \mathcal{U} \right) \right| \leq \frac{\varepsilon}{|\beta|} \right\}.\end{aligned}$$

Then, $\lambda \in \mathrm{Tr}_\varepsilon(\beta\mathcal{U} + \alpha\mathcal{V}, \mathcal{V})$. Thus, $\frac{\lambda - \alpha}{\beta} \in \mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U}, \mathcal{V})$. Hence, $\lambda \in \beta \mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U}, \mathcal{V}) + \alpha$.

(v) Let $\lambda \in \mathrm{Tr}_\varepsilon(\alpha\mathcal{V}, \mathcal{V})$, then

$$|\mathrm{Tr}(\lambda\mathcal{V} - \alpha\mathcal{V})| = |\lambda - \alpha| |\mathrm{Tr}(\mathcal{V})| \leq \varepsilon.$$

This means that $\mathrm{Tr}_\varepsilon(\alpha\mathcal{V}, \mathcal{V}) = \left\{ \lambda \in \mathbb{C} : |\lambda - \alpha| \leq \frac{\varepsilon}{|\mathrm{Tr}(\mathcal{V})|} \right\}$ for all $\lambda, \alpha \in \mathbb{C}$. Q.E.D.

Theorem 2.2. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,

(i) If $\mathcal{U} = \mathcal{Z}\mathcal{B}\mathcal{Z}^{-1}$ and $\mathcal{Z}\mathcal{V} = \mathcal{V}\mathcal{Z}$ for all nonsingular matrix $\mathcal{Z} \in \mathcal{M}_n(\mathbb{C})$ we have,

$$\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{B}, \mathcal{V}).$$

(ii) If $\mathcal{U} = \mathcal{Z}\mathcal{B}\mathcal{Z}^{-1}$ and $\mathcal{V} = \mathcal{Z}\mathcal{K}\mathcal{Z}^{-1}$ for all nonsingular matrix $\mathcal{Z} \in \mathcal{M}_n(\mathbb{C})$ we have,

$$\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{B}, \mathcal{K}).$$

(iii) The map $\mathbb{T} \rightarrow \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ is an upper semi continuous function from $\mathcal{M}_n(\mathbb{C})$ to compact subsets of $\mathbb{C}$. ◇

Proof. (i) Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{B}, \mathcal{V})$, then

$$\begin{aligned}|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{B})| &= |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{Z}^{-1}\mathcal{U}\mathcal{Z})|, \\ &= |\mathrm{Tr}(\lambda\mathcal{Z}^{-1}\mathcal{Z}\mathcal{V} - \mathcal{Z}^{-1}\mathcal{U}\mathcal{Z})| \\ &= |\mathrm{Tr}(\mathcal{Z}^{-1}(\lambda\mathcal{Z}\mathcal{V} - \mathcal{U}\mathcal{Z}))| \\ &= |\mathrm{Tr}(\mathcal{Z}^{-1}(\lambda\mathcal{V} - \mathcal{U})\mathcal{Z})| = |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon.\end{aligned}$$

It follows that, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$.

The proofs of items (ii) and (iii) follows immediately from Definition 1.1. Q.E.D.

The following example shows that the converse of the assertion (i) is not true.

Example 2.1. Let $\mathcal{U} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathcal{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $\mathcal{U}$ and $\mathcal{B}$ are not similar and for $\varepsilon > 0$, we have

$$\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{B}, \mathcal{V}) = \{\lambda \in \mathbb{C} : |\lambda - 2| \leq \varepsilon\}.$$

In the following, we obtain additional results on $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ that are useful in our analysis.

Theorem 2.3. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. Then, there is $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ such that $|\mathrm{Tr}(\mathcal{D})| \leq \varepsilon$ and $\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = 0$ if, and only if, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. $\diamond$

Proof. To see this, we suppose that there exists $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ such that $|\mathrm{Tr}(\mathcal{D})| \leq \varepsilon$ and

$$\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = 0.$$

Then,

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| = |\mathrm{Tr}(\mathcal{D})| \leq \varepsilon.$$

Thus, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. Conversely, let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. Then, we will discuss these two cases:

1st case: If $\lambda \in \mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$, then it is sufficient to take $(\mathcal{D} = 0_{n \times n})$.

2nd case: $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) \setminus \mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$. Then,

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon.$$

Now, we consider

$$\mathcal{D} = \frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n} \mathcal{I}.$$

It is easy to verify that, $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ and

$$|\mathrm{Tr}(\mathcal{D})| = \left| \mathrm{Tr} \left(\frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n} \mathcal{I} \right) \right| = \frac{|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})|}{n} \mathrm{Tr}(\mathcal{I}) \leq \varepsilon.$$

Also, we have

$$\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = \mathrm{Tr} \left(\lambda\mathcal{V} - \mathcal{U} - \frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n} \mathcal{I} \right) = 0.$$

Q.E.D.

Theorem 2.4. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,

$$\mathrm{Tr}_\delta(\mathcal{U}, \mathcal{V}) + \Theta_\varepsilon \subseteq \mathrm{Tr}_{\varepsilon+\delta}(\mathcal{U}, \mathcal{V}), \quad (1)$$

holds for $\delta, \varepsilon > 0$ with Θ_ε , denoting the closed disk in the complex plane centered at the origin with radius $\frac{\varepsilon}{|\mathrm{Tr}(\mathcal{V})|}$. If we take $\delta = 0$, we obtain an inner bound for $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$, namely

$$\mathrm{Tr}_0(\mathcal{U}, \mathcal{V}) + \Theta_\varepsilon \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}). \quad (2)$$

Proof. Let $\lambda \in \text{Tr}_\delta(\mathcal{U}, \mathcal{V}) + \Theta_\varepsilon$. Then, there exists there exists $\lambda_1 \in \text{Tr}_\delta(\mathcal{U}, \mathcal{V})$ and $\lambda_2 \in \Theta_\varepsilon$ such that $\lambda = \lambda_1 + \lambda_2$. Therefore,

$$|\text{Tr}(\lambda_1 \mathcal{V} - \mathcal{U})| \leq \delta$$

and

$$\begin{aligned} |\text{Tr}(\lambda \mathcal{V} - \mathcal{U})| &= |\text{Tr}((\lambda_1 + \lambda_2) \mathcal{V} - \mathcal{U})| \\ &= |\text{Tr}(\lambda_2 \mathcal{V}) + \text{Tr}(\lambda_1 \mathcal{V} - \mathcal{U})| \\ &\leq |\lambda_2| |\text{Tr}(\mathcal{V})| + |\text{Tr}(\lambda_1 \mathcal{V} - \mathcal{U})| \\ &\leq |\text{Tr}(\mathcal{V})| |\lambda_2| + |\text{Tr}(\lambda_1 \mathcal{V} - \mathcal{U})| \leq \varepsilon + \delta, \end{aligned}$$

so that (1) holds. Finally, let $\delta = 0$, then the desired inclusion (2) is obtained.

Q.E.D.

Theorem 2.5. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ such that $\mathcal{U}\mathcal{B} = \mathcal{B}\mathcal{U}$ and $\varepsilon > 0$. If $\mathcal{U}$ is normal, then

$$\text{Tr}_\varepsilon(\mathcal{U} + \mathcal{B}, \mathcal{V}) \subseteq \sigma(\mathcal{U}, \mathcal{V}) + \text{Tr}_\varepsilon(\mathcal{B}, \mathcal{V}).$$

Proof. We assume that $\mathcal{U}$ is normal, so there exists a unitary matrix $\mathcal{Z} \in \mathcal{M}_n(\mathbb{C})$ such that

$$\mathcal{Z}^* \mathcal{U} \mathcal{Z} = \lambda_1 \mathcal{I}_{n_1} \oplus \lambda_2 \mathcal{I}_{n_2} \oplus \dots \oplus \lambda_k \mathcal{I}_{n_k}.$$

The condition $\mathcal{U}\mathcal{B} = \mathcal{B}\mathcal{U}$ implies that

$$\mathcal{Z}^* \mathcal{B} \mathcal{Z} = \mathcal{U}_1 \oplus \mathcal{U}_2 \dots \oplus \mathcal{U}_k$$

where, $\mathcal{U}_i \in \mathcal{M}_{n_i}(\mathbb{C})$, $i = 1, \dots, k$. Then,

$$\begin{aligned} \text{Tr}_\varepsilon(\mathcal{U} + \mathcal{B}, \mathcal{V}) &= \text{Tr}_\varepsilon(\mathcal{Z}^* \mathcal{U} \mathcal{Z} + \mathcal{Z}^* \mathcal{B} \mathcal{Z}, \mathcal{V}) \\ &= \text{Tr}_\varepsilon((\lambda_1 \mathcal{I}_{n_1} + \mathcal{U}_1) \oplus \dots \oplus (\lambda_k \mathcal{I}_{n_k} + \mathcal{U}_k), \mathcal{V}) \\ &= \bigcup_{i=1}^k \text{Tr}_\varepsilon(\lambda_i \mathcal{I}_{n_i} + \mathcal{U}_i, \mathcal{V}) \\ &= \bigcup_{i=1}^k \lambda_i + \text{Tr}_\varepsilon(\mathcal{U}_i, \mathcal{V}) \\ &\subseteq \sigma(\mathcal{U}, \mathcal{V}) + \text{Tr}_\varepsilon(\mathcal{B}, \mathcal{V}). \end{aligned}$$

The proof is thus complete.

Q.E.D.

Remark 2.2. Let $\mathcal{U}, \mathcal{B}$ and $\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then, using Theorem 2.5, we obtain the following inequality,

$$\text{Trr}_\varepsilon(\mathcal{U} + \mathcal{B}, \mathcal{V}) \subseteq \text{r}(\mathcal{U}, \mathcal{V}) + \text{Trr}_\varepsilon(\mathcal{B}, \mathcal{V}).$$

◇

Theorem 2.6. Let $\mathcal{U}, \mathcal{B}$ and $\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,

$$(i) \operatorname{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}) = \operatorname{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V}).$$

$$(ii) \operatorname{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) + \operatorname{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V}) \subseteq \operatorname{Tr}_\varepsilon(\mathcal{U} + \mathcal{B}, \mathcal{V}).$$

Proof. (i) Let $\lambda \in \operatorname{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V})$, then

$$\begin{aligned} \varepsilon \geq |\operatorname{Tr}(\lambda\mathcal{V} - \mathcal{U}\mathcal{B})| &= |\operatorname{Tr}(\lambda\mathcal{V}) + \operatorname{Tr}(-\mathcal{U}\mathcal{B})| \\ &= |\operatorname{Tr}(\lambda\mathcal{V}) + \operatorname{Tr}(-\mathcal{B}\mathcal{U})| \\ &= |\operatorname{Tr}(\lambda\mathcal{V} - \mathcal{B}\mathcal{U})|. \end{aligned}$$

Hence, $\lambda \in \operatorname{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V})$. Thus,

$$\operatorname{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}) \subseteq \operatorname{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V}).$$

The conclusion can be obtained similarly to the first inclusion, then we deduce that

$$\operatorname{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V}) = \operatorname{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}).$$

(ii) Let $\lambda \in \operatorname{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) + \operatorname{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V})$. Then, there exists

$$\lambda_1 \in \operatorname{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) \text{ and } \lambda_2 \in \operatorname{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V})$$

such that $\lambda = \lambda_1 + \lambda_2$. Therefore,

$$\operatorname{Tr}(\lambda_1\mathcal{V} - \mathcal{U}) \leq \frac{\varepsilon}{2} \text{ and } \operatorname{Tr}(\lambda_2\mathcal{V} - \mathcal{B}) \leq \frac{\varepsilon}{2}.$$

On the other hand,

$$\begin{aligned} |\operatorname{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{B})| &= |\operatorname{Tr}(\lambda_1\mathcal{V} - \mathcal{U} + \lambda_2\mathcal{V} - \mathcal{B})| \\ &\leq |\operatorname{Tr}(\lambda_1\mathcal{V} - \mathcal{U})| + |\operatorname{Tr}(\lambda_2\mathcal{V} - \mathcal{B})| \\ &\leq \varepsilon \end{aligned}$$

Then, $\lambda \in \operatorname{Tr}_\varepsilon(\mathcal{U} + \mathcal{B}, \mathcal{V})$.

Q.E.D.

Theorem 2.7. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\mathcal{N} \in \mathcal{M}_n(\mathbb{C})$ is a nilpotent matrix and $\varepsilon > 0$. Then,

$$\operatorname{Tr}_\varepsilon(\mathcal{U} + \mathcal{N}, \mathcal{V}) = \operatorname{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

◇

Proof. " $\subseteq$ " Let $\lambda \in \operatorname{Tr}_\varepsilon(\mathcal{U} + \mathcal{N}, \mathcal{V})$, then $|\operatorname{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N})| \leq \varepsilon$. Since

$$|\operatorname{Tr}(\lambda\mathcal{V} - \mathcal{U}) - \operatorname{Tr}(\mathcal{N})| \leq \varepsilon.$$

Using the fact that the matrix trace vanishes on nilpotent matrices, therefore

$$\lambda \in \text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

Hence,

$$\text{Tr}_\varepsilon(\mathcal{U} + \mathcal{N}, \mathcal{V}) \subseteq \text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

" $\supseteq$ " Let $\lambda \in \text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$, then $|\text{Tr}(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon$. Now, we can write for any $\lambda \in \mathbb{C}$

$$|\text{Tr}(\lambda\mathcal{V} - \mathcal{U})| = |\text{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N} + \mathcal{N})| = |\text{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N}) + \text{Tr}(\mathcal{N})|.$$

Because, $\text{Tr}(\mathcal{N}) = 0$, it follows that $|\text{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N})| \leq \varepsilon$. Consequently,

$$\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) \subseteq \text{Tr}_\varepsilon(\mathcal{U} + \mathcal{N}, \mathcal{V}).$$

Q.E.D.

3 Trace pseudospectral mapping Theorem

Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and f be an analytic function defined on D , an open set containing $\text{Tr}_0(\mathcal{U}, \mathcal{V})$. For each $\varepsilon > 0$, we define

$$\varphi(\varepsilon) = \sup_{\lambda \in \text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})} |\text{Tr}(f(\lambda)\mathcal{V} - f(\mathcal{U}))|$$

and suppose there exists $\varepsilon_0 > 0$ such that $\text{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D)$. Then, for $0 < \varepsilon < \varepsilon_0$ we define

$$\phi(\varepsilon) = \sup_{\mu \in f^{-1}(\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \cap D} |\text{Tr}(\mu\mathcal{V} - \mathcal{U})|.$$

Lemma 3.1. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, then $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are well defined, $\lim_{\varepsilon \rightarrow 0} \varphi(\varepsilon) = 0$ and $\lim_{\varepsilon \rightarrow 0} \phi(\varepsilon) = 0$.*

Proof. In the order to prove that $\varphi(\varepsilon)$ is well defined, we define $h : \mathbb{C} \rightarrow \mathbb{R}_+$

$$h(\lambda) = |\text{Tr}(f(\lambda)\mathcal{V} - f(\mathcal{U}))|$$

Since $h(\lambda)$ is continuous and $\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ is a compact subset of $\mathbb{C}$, then it is clear that

$$\varphi(\varepsilon) = \sup \{h(\lambda) : \lambda \in \text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})\}.$$

We conclude, $\varphi(\varepsilon)$ is well defined. Now, let assume that there exists $\varepsilon_0 > 0$ such that

$$\text{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D).$$

We show that for $0 < \varepsilon < \varepsilon_0$, $\phi(\varepsilon)$ is well defined. Define $g : \mathbb{C} \rightarrow \mathbb{R}_+$,

$$g(\mu) = |\text{Tr}(\mu\mathcal{V} - \mathcal{U})|.$$

Since g is continuous for all $\mu \in \mathbb{C}$, then $\phi(\varepsilon)$ is well defined. It is also clear that $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are a monotonically non-decreasing function, $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ goes to zero as ε goes to zero. Q.E.D.

Theorem 3.1. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on D , an open set containing $\text{Tr}_0(\mathcal{U}, \mathcal{V})$. Then, for each*

$$f(\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \subseteq \text{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V}),$$

where $\varphi(\varepsilon)$ defined above.

Proof. Let $\lambda \in \text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. Then, using Lemma 3.1 we obtain that $\varphi(\varepsilon)$ is well defined and $\lim_{\varepsilon \rightarrow 0} \varphi(\varepsilon) = 0$. Therefore, $h(\lambda) \leq \varphi(\varepsilon)$. Hence

$$|\text{Tr}(f(\lambda)\mathcal{V} - f(\mathcal{U}))| := h(\lambda) \leq \varphi(\varepsilon).$$

Thus, $f(\lambda) \in \text{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V})$. This means that

$$f(\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \subseteq \text{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V}).$$

Q.E.D.

Theorem 3.2. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on D , an open set containing $\text{Tr}_0(\mathcal{U}, \mathcal{V})$. Then, for each*

$$\text{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq f(\text{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})).$$

where $\phi(\varepsilon)$ defined above.

Proof. Let $\lambda \in \text{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V})$. Then, using Lemma 3.1 we obtain the existence of $\varepsilon_0 > 0$ such that

$$\text{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq \text{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D).$$

Consider $\mu \in D$ such that $\lambda = f(\mu)$. Then $\mu \in f^{-1}(\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}))$, hence

$$g(\mu) \leq \phi(\varepsilon).$$

Therefore,

$$|\text{Tr}(\mu\mathcal{V} - \mathcal{U})| := g(\mu) \leq \phi(\varepsilon)$$

Thus, $\mu \in \text{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})$. Then, $\lambda = f(\mu) \in f(\text{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V}))$. This means that

$$\text{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq f(\text{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})).$$

Q.E.D.

Corollary 3.1. *Combining the two inclusions in Theorems 3.1 and 3.2, we get*

$$f(\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \subseteq \text{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V}) \subseteq f(\text{Tr}_{\phi(\varphi(\varepsilon))}(\mathcal{U}, \mathcal{V}))$$

and

$$\text{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq f(\text{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})) \subseteq \text{Tr}_{\varphi(\phi(\varepsilon))}(f(\mathcal{U}), \mathcal{V}).$$

Here are some remarks.

Remark 3.1. (i) *It will be clear from the proofs of Theorems 3.1 and 3.2 that the functions φ and ϕ measure the sizes of the trace pseudo-spectra are optimal.*

(ii) *From the definitions of φ and ϕ , the set inclusions are sharp in the sense that the functions cannot be replaced by smaller functions.*

(iii) *In general, the spectral mapping theorem is not true for generalized trace pseudo-spectrum.*

Example 3.1. *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta \neq 0$ and let $\mathcal{U} = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$, $\mathcal{V} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $f(\lambda) = \lambda^2$. Then $f(\mathcal{U}) = \begin{pmatrix} \alpha^2 & \alpha + \beta \\ 0 & \beta^2 \end{pmatrix}$. A direct computation shows that*

$$\text{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) = \{\lambda \in \mathbb{C} : |2\lambda - \alpha^2| \leq \varepsilon - \beta^2\},$$

$$f(\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) = \{\lambda^2 \in \mathbb{C} : |2\lambda - \alpha^2| \leq \varepsilon - \beta^2\}.$$

We can see for all $\varepsilon > 0$ that $\text{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \neq f(\text{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}))$.

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