Totally umbilical proper slant submanifolds of para-Kenmotsu manifold

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ABSTRACT

In this paper, we study slant submanifolds of a para-Kenmotsu manifold. We prove that totally umbilical slant submanifold of a para-Kenmotsu manifold is either invariant or anti-invariant or dimension of submanifold is 1 or the mean curvature vector H of the submanifold lies in the invariant normal subbundle.

RESUMEN

En este paper estudiamos subvariedades inclinadas en variedades para-Kenmotsu. Demostramos que una subvariedad inclinada en una variedad para-Kenmotsu totalmente umbilical es invariante, o anti-invariante, o una subvariedad de dimensión 1, o el vector de curvatura media H de la subvariedad vive en el fibrado normal invariante.

Keywords and Phrases: para-Kenmotsu manifold; totally umbilical; slant submanifold.

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1 Introduction

The study of submanifolds of an almost contact manifold is one of the utmost interesting topics in differential geometry. According to the behaviour of the tangent bundle of a submanifold with respect to action of the almost contact structure ϕ of the ambient manifold, there are two well known classes of submanifolds, namely, invariant and anti-invariant submanifolds. Chen [4], introduced the notion of slant submanifolds of the almost Hermitian manifolds. The contact version of slant submanifolds were given by Lotta [12]. Since then many research articles have been appeared on the existence of different contact and lorentzian manifolds (See. [1, 3, 7, 14, 15]).

Motivated by the above studies, in the present paper we study slant submanifolds of almost para-Kenmotsu manifold and give a classification of results. Also we prove that totally umbilical slant submanifolds of para-Kenmotsu manifolds are totally geodesic.

The paper is organized as follows: In section 2, we review some basic concepts of para-Kenmotsu manifold and submanifold theory. Section 3 is the main section of this paper. In this section we give the classification result of totally umbilical slant submanifolds of para-Kenmotsu manifold. Further, we prove that totally umbilical slant submanifolds of a para-Kenmotsu manifold are totally geodesic.

2 Preliminaries

Let $\tilde{M}$ be a $(2m+1)$ -dimensional smooth manifold, ϕ a tensor field of type $(1, 1)$, ξ a vector field and η a 1-form. We say that (ϕ, ξ, η) is an almost para contact structure on $\tilde{M}$ if [18]

$$\phi\xi = 0, \quad \eta \cdot \phi = 0, \quad \text{rank}(\phi) = 2m, \quad (2.1)$$

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1. \quad (2.2)$$

If an almost paracontact manifold admits a pseudo Riemannian metric g of signature $(m+1, m)$ satisfying

$$g(\phi\cdot, \phi\cdot) = -g + \eta \otimes \eta \quad (2.3)$$

called almost para contact metric manifold. Examples of almost para contact metric structure are given in [6] and [9].

Analogous to the definition of Kenmotsu manifold [10], Welyczko [17] introduced para-Kenmotsu structure for three dimensional normal almost para contact metric structures. The similar notion called p-Kenmotsu structure appears in the Sinha and Sai Prasad [16].

Definition 2.1. *An almost para contact metric manifold $M(\phi, \xi, \eta, g)$ is para-Kenmotsu manifold if the Levi-Civita connection $\tilde{\nabla}$ of g satisfies*

$$(\tilde{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (2.4)$$

for any $X, Y \in \chi(M)$, (where $\chi(M)$ is the set of all differential vector fields on M).

From (2.4), we have

$$\tilde{\nabla}_X \xi = X - \eta(X)\xi. \quad (2.5)$$

Assume M is a submanifold of a para-Kenmotsu manifold $\tilde{M}$. Let g and ∇ be the induced Riemannian metric and connections of M , respectively. Then the Gauss and Weingarten formulae are given respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (2.6)$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.7)$$

for all X, Y on TM and $N \in T^\perp M$, where $\nabla^\perp$ is the normal connection and A is the shape operator of M with respect to the unit normal vector N . The second fundamental form σ and the shape operator A are related by:

$$g(\sigma(X, Y), N) = g(A_N X, Y). \quad (2.8)$$

Now for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we write

$$\phi X = PX + FX, \quad (2.9)$$

$$\phi V = pV + fV. \quad (2.10)$$

For $X, Y \in \Gamma(TM)$, it is easy to observe from (2.1) and (2.9) that

$$g(PX, Y) = -g(X, PY). \quad (2.11)$$

The covariant derivatives of the endomorphisms ϕ , P and F are defined respectively as

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y, \quad \forall X, Y \in \Gamma(T\tilde{M}), \quad (2.12)$$

$$(\tilde{\nabla}_X P)Y = \nabla_X PY - P \nabla_X Y, \quad \forall X, Y \in \Gamma(TM), \quad (2.13)$$

$$(\tilde{\nabla}_X F)Y = \nabla_X FY - F \nabla_X Y, \quad \forall X, Y \in \Gamma(TM). \quad (2.14)$$

The structure vector field ξ has been considered to be tangential to M throughout this paper, else M is simply anti-invariant [12]. Since $\xi \in TM$, for any $X \in \Gamma(TM)$ by virtue of (2.5) and (2.6), we have

$$\nabla_X \xi = X - \eta(X)\xi \quad \text{and} \quad \sigma(X, \xi) = 0. \quad (2.15)$$

Making use of (2.4), (2.6), (2.7), (2.9), (2.10) and (2.12)-(2.14), we obtain

$$(\tilde{\nabla}_X P)Y = p\sigma(X, Y) + A_{FY}X + g(PX, Y)\xi - \eta(Y)PX, \quad (2.16)$$

$$(\tilde{\nabla}_X F)Y = f\sigma(X, Y) - \sigma(X, PY) - \eta(Y)FX. \quad (2.17)$$

A submanifold M of an almost para contact metric manifold $\tilde{M}$ is said to be totally umbilical if

$$\sigma(X, Y) = g(X, Y)H, \quad (2.18)$$

where H is the mean curvature vector of M . Further M is totally geodesic if $\sigma(X, Y) = 0$ and minimal if $H = 0$.

3 Slant submanifolds of an almost contact metric manifold

For any $x \in M$ and $X \in T_x M$ such that X, ξ are linearly independent, the angle $\theta(x) \in [0, \frac{\pi}{2}]$ between ϕX and $T_x M$ is a constant θ , that is θ does not depend on the choice of X and $x \in M$. θ is called the slant angle of M in $\tilde{M}$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively [5]. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

We mention the following results for later use.

Theorem 3.1. [1] Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = -\lambda(I - \eta \otimes \xi). \quad (3.1)$$

Further more, if θ is the slant angle of M , then $\lambda = \cos^2 \theta$.

Corollary 1. [1] Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ . Then, for any $X, Y \in TM$, we have

$$g(PX, PY) = -\cos^2 \theta (g(X, Y) - \eta(X)\eta(Y)), \quad (3.2)$$

$$g(FX, FY) = -\sin^2 \theta (g(X, Y) - \eta(X)\eta(Y)). \quad (3.3)$$

Theorem 3.2. Let M be a totally umbilical slant submanifold of a para-Kenmotsu manifold $\tilde{M}$. Then either one of the following statements is true:

- (i) M is invariant;
 - (ii) M is anti-invariant;
 - (iii) M is totally geodesic;
 - (iv) $\dim M = 1$;
 - (v) If M is proper slant, then $H \in \Gamma(\mu)$;
- where H is the mean curvature vector of M .

Proof. Suppose M is totally umbilical slant submanifold, then we have

$$\sigma(PX, PX) = g(PX, PX)H = \cos^2 \theta \{-\|X\|^2 + \eta^2(X)\}H.$$

By virtue of (2.6), one can get

$$\cos^2 \theta \{-\|X\|^2 + \eta^2(X)\}H = \tilde{\nabla}_{PX} PX - \nabla_{PX} PX.$$

From (2.9), we have

$$\cos^2 \theta \{-\|X\|^2 + \eta^2(X)\}H = \tilde{\nabla}_{PX} \phi X - \tilde{\nabla}_{PX} FX - \nabla_{PX} PX.$$

Applying (2.7) and (2.12), we get

$$\cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H = (\tilde{\nabla}_{PX}\phi)X + \phi\tilde{\nabla}_{PX}X + A_{FX}PX - \nabla_{PX}^\perp FX - \nabla_{PX}PX.$$

Using (2.4) and (2.6), we obtain

$$\begin{aligned} \cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H &= g(\phi PX, X)\xi - \eta(X)\phi PX + \phi(\nabla_{PX}X + \sigma(X, PX)) \\ &\quad + A_{FX}PX - \nabla_{PX}^\perp FX - \nabla_{PX}PX. \end{aligned}$$

From (2.9), (2.11), (2.18) and the fact that X and PX are orthogonal vector fields on M , we arrive at

$$\begin{aligned} \cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H &= -g(PX, PX)\xi - \eta(X)P^2X - \eta(X)FPX + P\nabla_{PX}X + F\nabla_{PX}X \\ &\quad + A_{FX}PX - \nabla_{PX}^\perp FX - \nabla_{PX}PX. \end{aligned}$$

Then applying (3.1) and (3.2), we obtain

$$\begin{aligned} \cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H &= \cos^2\theta\{\|X\|^2 - \eta^2(X)\}\xi + \cos^2\theta\eta(X)\{X - \eta(X)\}\xi - \eta(X)FPX \\ &\quad + P\nabla_{PX}X + F\nabla_{PX}X + A_{FX}PX - \nabla_{PX}^\perp FX - \nabla_{PX}PX. \end{aligned} \quad (3.4)$$

Taking inner product with PX in (3.4), we get

$$0 = g(P\nabla_{PX}X, PX) + g(A_{FX}PX, PX) - g(\nabla_{PX}PX, PX). \quad (3.5)$$

By virtue of (3.2), the first term of (3.5) can be written as

$$g(P\nabla_{PX}X, PX) = -\cos^2\theta\{g(\nabla_{PX}X, X) - \eta(X)g(\nabla_{PX}X, \xi)\}. \quad (3.6)$$

We simplify the third term of (3.5) as follows

$$\begin{aligned} g(\nabla_{PX}PX, PX) &= g(\tilde{\nabla}_{PX}PX, PX) = \frac{1}{2}PXg(PX, PX). \\ &= \frac{1}{2}PX[-\cos^2\theta\{(g(X, X) - \eta^2(X))\}] \\ &= -\frac{1}{2}\cos^2\theta[PXg(X, X) - P(X)(g(X, \xi)g(X, \xi))] \\ &= -\frac{1}{2}\cos^2\theta[PXg(X, X) - 2\eta(X)P(X)g(X, \xi)] \\ &= -\frac{1}{2}\cos^2\theta[2g(\tilde{\nabla}_{PX}X, X) - 2\eta(X)\{g(\tilde{\nabla}_{PX}X, \xi) + g(X, \tilde{\nabla}_{PX}\xi)\}]. \end{aligned}$$

Using (2.5), (2.6), (3.6) and the fact that X and PX are orthogonal vector fields on M , we derive

$$\begin{aligned} g(\nabla_{PX}PX, PX) &= -\cos^2\theta[g(\nabla_{PX}X, X) - \eta(X)g(\nabla_{PX}X, \xi) \\ &\quad - \eta(X)g(X, PX - \eta(PX)\xi)] \\ &= -\cos^2\theta[g(\nabla_{PX}X, X) - \eta(X)g(\nabla_{PX}X, \xi)] \\ \rightarrow g(\nabla_{PX}PX, PX) &= g(P\nabla_{PX}X, PX). \end{aligned}$$

Using this fact in (3.5), we obtain

$$0 = g(A_{FX}PX, PX) = g(\sigma(PX, PX), FX).$$

As M is totally umbilical slant, then from (2.18) and (3.2), we obtain

$$0 = -\cos^2\theta\{\|X\|^2 - \eta^2(X)\}g(H, FX). \quad (3.7)$$

Thus from (3.7), we conclude that either $\theta = \frac{\pi}{2}$ that is M is anti-invariant which part (ii) or the vector field X is parallel to the structure vector field ξ , i.e., M is 1-dimensional submanifold which is fourth part of the theorem or $H \perp FX$, for all $X \in \Gamma(TM)$, i.e., $H \in \Gamma(\mu)$ which is the last part of the theorem or $H = 0$, i.e., M is totally geodesic which is (iii) or $FX = 0$, i.e., M is invariant which is part (i). This completes the proof of the theorem. $\square$

Theorem 3.3. *Every totally umbilical proper slant submanifold of a para-Kenmotsu manifold is totally geodesic.*

Proof. Let M be a totally umbilical proper slant submanifold of a para-Kenmotsu manifold $\tilde{M}$, then for any $X, Y \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Using equations (2.6) and (2.9), we get

$$\tilde{\nabla}_X PY + \tilde{\nabla}_X FY - \phi(\nabla_X Y + \sigma(X, Y)) = g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX.$$

Again from (2.6), (2.7) and (2.9), we obtain

$$\begin{aligned} g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX &= \nabla_X PY + \sigma(X, PY) - A_{FY}X \\ &\quad + \nabla_X^\perp FY - P\nabla_X Y - F\nabla_X Y - \phi\sigma(X, Y). \end{aligned}$$

As M is totally umbilical, then

$$\begin{aligned} g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX &= \nabla_X PY + g(X, PY)H - A_{FY}X + \nabla_X^\perp FY \\ &\quad - P\nabla_X Y - F\nabla_X Y - g(X, Y)\phi H. \end{aligned} \quad (3.8)$$

Taking the inner product with ϕH in (3.8) and from the fact that $H \in \Gamma(\mu)$, we obtain

$$g(\nabla_X^\perp FY, \phi H) = -g(X, Y)\|H\|^2.$$

Applying (2.7) and the property of Riemannian connection the above equation takes the form

$$g(FY, \nabla_X^\perp \phi H) = g(X, Y)\|H\|^2. \quad (3.9)$$

Now for any $X \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X \phi H = (\tilde{\nabla}_X \phi)H + \phi \tilde{\nabla}_X H.$$

Using the fact $H \in \Gamma(\mu)$ and by virtue of (2.4), (2.7) and (2.9), we obtain

$$-A_{\phi H}X + \nabla_X^\perp \phi H = -PA_HX - FA_HX + \phi \nabla_X^\perp H. \quad (3.10)$$

Also for any $X \in \Gamma(TM)$, we have

$$g(\nabla_X^\perp H, FX) = g(\tilde{\nabla}_X H, FX) = -g(H, \tilde{\nabla}_X FX).$$

Using (2.9), we obtain

$$g(\nabla_X^\perp H, FX) = -g(H, \tilde{\nabla}_X \phi X) + g(H, \tilde{\nabla}_X PX).$$

Further from (2.6) and (2.12), we derive

$$g(\nabla_X^\perp H, FX) = -g(H, (\tilde{\nabla}_X \phi)X) - g(H, \phi \tilde{\nabla}_X X) + g(H, \sigma(X, PX)).$$

Using (2.4) and (2.18), the first and last term of right hand side of the above equation are identically zero and hence by (2.3), the second term gives

$$g(\nabla_X^\perp H, FX) = g(\phi H, \tilde{\nabla}_X X).$$

Again by using (2.6) and (2.18), we obtain

$$g(\nabla_X^\perp H, FX) = g(\phi H, H)\|X\|^2 = 0.$$

This means that

$$\nabla_X^\perp H \in \Gamma(\mu). \quad (3.11)$$

Now, taking the inner product in (3.10) with FY for any $Y \in \Gamma(TM)$, we get

$$g(\nabla_X^\perp \phi H, FY) = -g(FA_HX, FY) + g(\phi \nabla_X^\perp H, FY).$$

Using (3.11) and from (3.3) and (3.9), we obtain

$$g(X, Y)\|H\|^2 = \sin^2 \theta \{g(A_HX, Y) - \eta(Y)g(A_HX, \xi)\}. \quad (3.12)$$

Hence by (2.8) and (2.18), the above equation reduces to

$$g(X, Y)\|H\|^2 = \sin^2 \theta \{g(X, Y)\|H\|^2 - \eta(Y)g(\sigma(X, Y), H)\}. \quad (3.13)$$

Since for a para-Kenmotsu manifold $\tilde{M}$, $\sigma(X, \xi) = 0$ for any X tangent to $\tilde{M}$, thus we obtain

$$g(X, Y)\|H\|^2 = \sin^2 \theta \{g(X, Y)\|H\|^2\}.$$

Therefore, the above equation can be written as

$$\cos^2 \theta g(X, Y)\|H\|^2 = 0. \quad (3.14)$$

Since M is proper slant submanifold, thus from (3.14) we conclude that $H = 0$, i.e., M is totally geodesic in $\tilde{M}$. This completes the proof. $\square$

References

- [1] A.M. Blaga, *Invariant, anti-invariant and slant submanifolds of a para-Kenmotsu manifold*, BSG Proceedings, 24 (2017), 19-28.
- [2] A.M. Blaga, *Eta-Ricci solitons on para-Kenmotsu manifolds*, Balkan Journal of Geometry and Its Applications, 20(1) (2015), 1-13.
- [3] J.L. Cabrerizo, A. Carriazo and L.M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasgow Math. J., 42 (2000), 125-138.
- [4] B.Y. Chen, *Slant immersions*, Bull. Aust. Math. Soc., 41 (1990), 135-147.
- [5] B.Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, (1990).
- [6] P. Dacko and Z. Olszak, *On weakly para-cosymplectic manifolds of dimension 3*, J. Geom. Phys., 57 (2007), 561-570.
- [7] R.S. Gupta, S.M. Khursheed Haider and M.H. Shahid, *Slant submanifolds of a Kenmotsu manifold*, Radovi Matematicki, Vol. 12 (2004), 205-214.
- [8] R.S. Gupta and P.K. Pandey, *Structure on a slant submanifold of a Kenmotsu manifold*, Differential Geometry - Dynamical Systems, 10 (2008), 139-147.
- [9] S. Ivanov, D. Vassilev and S. Zamkovoy, *Conformal paracontact curvature and the local flatness theorem*, Geom. Dedicata, 144 (2010), 79-100.
- [10] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24 (1972), 93-103.
- [11] M.A. Khan, S. Uddin and K. Singh, *A classification on totally umbilical proper slant and hemi-slant submanifolds of a nearly trans-Sasakian manifold*, Differential Geometry - Dynamical Systems, 13 (2011), 117-127.
- [12] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roum., 39 (1996), 183-198.
- [13] A. Lotta, *Three dimensional slant submanifolds of K-contact manifolds*, Balkan J. Geom. Appl., 3(1) (1998), 37-51.
- [14] M.S. Siddesha and C.S. Bagewadi, *On slant submanifolds of (k, μ) -contact manifold*, Differential Geometry-Dynamical Systems, 18 (2016), 123-131.
- [15] M.S. Siddesha and C.S. Bagewadi, *Semi-slant submanifolds of (k, μ) -contact manifold*, Commun. Fac. Sci. Univ. Ser. A₁ Math. Stat., 67(2) (2017), 116-125.

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- [16] B.B. Sinha and K.L. Sai Prasad, *A class of almost para contact metric manifolds*, Bull. Cal. Math. Soc., 87 (1995), 307-312.
 - [17] J. Welyczko, *Slant curves in 3-dimensional normal almost paracontact metric manifolds*, Mediterr. J. Math., DOI 10.1007/s00009-013-0361-2, 2013.
 - [18] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom., 36(1) (2008), 37-60.