Caputo fractional Iyengar type Inequalities

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ABSTRACT

Here we present Caputo fractional Iyengar type inequalities with respect to L_p norms, with $1 \leq p \leq \infty$. The method is based on the right and left Caputo fractional Taylor's formulae.

RESUMEN

Aquí presentamos desigualdades de tipo Caputo fraccional Iyengar con respecto a las normas L_p , con $1 \leq p \leq \infty$. El método se basa en las fórmulas de Taylor fraccionales de Caputo derecha e izquierda.

Keywords and Phrases: Iyengar inequality, right and left Caputo fractional, Taylor formulae, Caputo fractional derivative.

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1 Introduction

We are motivated by the following famous Iyengar inequality (1938), [4].

Theorem 1. *Let f be a differentiable function on $[a, b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x) dx - \frac{1}{2} (b-a) (f(a) + f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b) - f(a))^2}{4M}. \quad (1)$$

We need

Definition 2. ([1], p. 394) Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ the ceiling of the number), $f \in AC^n([a, b])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[a, b]$). The left Caputo fractional derivative of order ν is defined as

$$D_{*a}^\nu f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) dt, \quad (2)$$

$\forall x \in [a, b]$, and it exists almost everywhere over $[a, b]$.

We need

Definition 3. ([2], p. 336-337) Let $\nu > 0$, $n = \lceil \nu \rceil$, $f \in AC^n([a, b])$. The right Caputo fractional derivative of order ν is defined as

$$D_{b-}^\nu f(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (z-x)^{n-\nu-1} f^{(n)}(z) dz, \quad (3)$$

$\forall x \in [a, b]$, and exists almost everywhere over $[a, b]$.

2 Main Results

We present the following Caputo fractional Iyengar type inequalities:

Theorem 4. Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), and $f \in AC^n([a, b])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[a, b]$). We assume that $D_{*a}^\nu f, D_{b-}^\nu f \in L_\infty([a, b])$. Then

i)

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq \frac{\max \left\{ \|D_{*a}^\nu f\|_{L_\infty([a, b])}, \|D_{b-}^\nu f\|_{L_\infty([a, b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \quad (4)$$

$\forall t \in [a, b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (4) is minimized, and we get:

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\max \left\{ \|D_{*a}^\nu f\|_{L_\infty([a,b])}, \|D_{b-}^\nu f\|_{L_\infty([a,b])} \right\} (b-a)^{\nu+1}}{\Gamma(\nu+2) 2^\nu}, \quad (5)$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, \dots, n-1$, we obtain

$$\left| \int_a^b f(x) dx \right| \leq \frac{\max \left\{ \|D_{*a}^\nu f\|_{L_\infty([a,b])}, \|D_{b-}^\nu f\|_{L_\infty([a,b])} \right\} (b-a)^{\nu+1}}{\Gamma(\nu+2) 2^\nu}, \quad (6)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, \dots, N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq \frac{\max \left\{ \|D_{*a}^\nu f\|_{L_\infty([a,b])}, \|D_{b-}^\nu f\|_{L_\infty([a,b])} \right\} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right]}{\Gamma(\nu+2)}, \quad (7)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \dots, n-1$, from (7) we get:

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq \frac{\max \left\{ \|D_{*a}^\nu f\|_{L_\infty([a,b])}, \|D_{b-}^\nu f\|_{L_\infty([a,b])} \right\} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right]}{\Gamma(\nu+2)}, \quad (8)$$

$j = 0, 1, 2, \dots, N$,

vi) when $N = 2$ and $j = 1$, (8) turns to

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\max \left\{ \|D_{*a}^\nu f\|_{L_\infty([a,b])}, \|D_{b-}^\nu f\|_{L_\infty([a,b])} \right\} (b-a)^{\nu+1}}{\Gamma(\nu+2) 2^\nu}, \quad (9)$$

vii) when $0 < \nu \leq 1$, inequality (9) is again valid without any boundary conditions.

Proof. Let $\nu > 0$, $n = \lceil \nu \rceil$, and $f \in AC^n([a, b])$. Then by ([3], p. 54) left Caputo fractional Taylor's formula we have

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{*a}^\nu f(t) dt, \quad (10)$$

$\forall x \in [a, b]$.

Also by ([2], p. 341) right Caputo fractional Taylor's formula we get:

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k = \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} D_{b-}^\nu f(z) dz, \quad (11)$$

$\forall x \in [a, b]$.

By (10) we derive

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\|D_{*a}^\nu f\|_{L_\infty([a,b])}}{\Gamma(\nu+1)} (x-a)^\nu, \quad (12)$$

and by (11) we obtain

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\|D_{b-}^\nu f\|_{L_\infty([a,b])}}{\Gamma(\nu+1)} (b-x)^\nu, \quad (13)$$

$\forall x \in [a, b]$.

Call

$$\gamma_1 := \frac{\|D_{*a}^\nu f\|_{L_\infty([a,b])}}{\Gamma(\nu+1)}, \quad (14)$$

and

$$\gamma_2 := \frac{\|D_{b-}^\nu f\|_{L_\infty([a,b])}}{\Gamma(\nu+1)}. \quad (15)$$

Set

$$\gamma := \max(\gamma_1, \gamma_2). \quad (16)$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \gamma (x-a)^\nu, \quad (17)$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \gamma (b-x)^\nu, \quad (18)$$

$\forall x \in [a, b]$.

Hence it holds

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k - \gamma (x-a)^\gamma \leq f(x) \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + \gamma (x-a)^\gamma \quad (19)$$

and

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k - \gamma (b-x)^\gamma \leq f(x) \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \gamma (b-x)^\gamma, \quad (20)$$

$\forall x \in [a, b]$.

Let any $t \in [a, b]$, then by integration over $[a, t]$ and $[t, b]$, respectively, we obtain

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{(k+1)!} (t-a)^{k+1} - \frac{\gamma}{(\gamma+1)} (t-a)^{\gamma+1} &\leq \int_a^t f(x) dx \leq \\ \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{(k+1)!} (t-a)^{k+1} + \frac{\gamma}{(\gamma+1)} (t-a)^{\gamma+1}, \end{aligned} \quad (21)$$

and

$$\begin{aligned} - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{(k+1)!} (t-b)^{k+1} - \frac{\gamma}{(\gamma+1)} (b-t)^{\gamma+1} &\leq \int_t^b f(x) dx \leq \\ - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{(k+1)!} (t-b)^{k+1} + \frac{\gamma}{(\gamma+1)} (b-t)^{\gamma+1}. \end{aligned} \quad (22)$$

Adding (21) and (22), we obtain

$$\begin{aligned} &\left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} - f^{(k)}(b) (t-b)^{k+1} \right] \right\} - \\ &\quad \frac{\gamma}{(\gamma+1)} \left[(t-a)^{\gamma+1} + (b-t)^{\gamma+1} \right] \leq \int_a^b f(x) dx \leq \\ &\left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} - f^{(k)}(b) (t-b)^{k+1} \right] \right\} + \\ &\quad \frac{\gamma}{(\gamma+1)} \left[(t-a)^{\gamma+1} + (b-t)^{\gamma+1} \right], \end{aligned} \quad (23)$$

$\forall t \in [a, b]$.

Consequently we derive:

$$\begin{aligned} \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq \\ \frac{\gamma}{(\gamma+1)} \left[(t-a)^{\gamma+1} + (b-t)^{\gamma+1} \right], \end{aligned} \quad (24)$$

$\forall t \in [a, b]$.

Let us consider

$$g(t) := (t-a)^{\nu+1} + (b-t)^{\nu+1}, \quad \forall t \in [a, b].$$

Hence

$$g'(t) = (\nu+1) [(t-a)^\nu - (b-t)^\nu] = 0,$$

giving $(t-a)^\nu = (b-t)^\nu$ and $t-a = b-t$, that is $t = \frac{a+b}{2}$ the only critical number here.

We have $g(a) = g(b) = (b-a)^{\nu+1}$, and $g\left(\frac{a+b}{2}\right) = \frac{(b-a)^{\nu+1}}{2^\nu}$, which the minimum of g over $[a, b]$.

Consequently the right hand side of (24) is minimized when $t = \frac{a+b}{2}$, with value $\frac{\gamma}{(\nu+1)} \frac{(b-a)^{\nu+1}}{2^\nu}$.

Assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, for $k = 0, 1, \dots, n-1$, then we obtain that

$$\left| \int_a^b f(x) dx \right| \leq \frac{\gamma}{(\nu+1)} \frac{(b-a)^{\nu+1}}{2^\nu}, \quad (25)$$

which is a sharp inequality.

When $t = \frac{a+b}{2}$, then (24) becomes

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\gamma}{(\nu+1)} \frac{(b-a)^{\nu+1}}{2^\nu}. \quad (26)$$

Next let $N \in \mathbb{N}$, $j = 0, 1, 2, \dots, N$ and $t_j = a + j \left(\frac{b-a}{N}\right)$, that is $t_0 = a$, $t_1 = a + \frac{b-a}{N}$, ..., $t_N = b$.

Hence it holds

$$t_j - a = j \left(\frac{b-a}{N}\right), \quad (b - t_j) = (N-j) \left(\frac{b-a}{N}\right), \quad j = 0, 1, 2, \dots, N. \quad (27)$$

We notice that

$$(t_j - a)^{\nu+1} + (b - t_j)^{\nu+1} = \left(\frac{b-a}{N}\right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \quad (28)$$

$j = 0, 1, 2, \dots, N$,

and (for $k = 0, 1, \dots, n-1$)

$$\begin{aligned} & \left[f^{(k)}(a) (t_j - a)^{k+1} + (-1)^k f^{(k)}(b) (b - t_j)^{k+1} \right] = \\ & \left[f^{(k)}(a) j^{k+1} \left(\frac{b-a}{N}\right)^{k+1} + (-1)^k f^{(k)}(b) (N-j)^{k+1} \left(\frac{b-a}{N}\right)^{k+1} \right] = \end{aligned}$$

$$\left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a) j^{k+1} + (-1)^k f^{(k)}(b) (N-j)^{k+1} \right], \quad (29)$$

$j = 0, 1, 2, \dots, N$.

By (24) we get

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a) j^{k+1} + (-1)^k f^{(k)}(b) (N-j)^{k+1} \right] \right|$$

$$\leq \frac{\gamma}{(\nu+1)} \left(\frac{b-a}{N}\right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \quad (30)$$

$j = 0, 1, 2, \dots, N$.

If $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \dots, n-1$, then (30) becomes

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{N}\right) [jf(a) + (N-j)f(b)] \right| \leq$$

$$\frac{\gamma}{(\nu+1)} \left(\frac{b-a}{N}\right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \quad (31)$$

$j = 0, 1, 2, \dots, N$.

When $N = 2$ and $j = 1$, then (31) becomes

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{2}\right) (f(a) + f(b)) \right| \leq$$

$$\frac{\gamma}{(\nu+1)} 2 \left(\frac{b-a}{2}\right)^{\nu+1} = \frac{\gamma}{(\nu+1)} \frac{(b-a)^{\nu+1}}{2^\nu}. \quad (32)$$

Let $0 < \nu \leq 1$, then $n = \lceil \nu \rceil = 1$. In that case, without any boundary conditions, we derive from (32) again that

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{2}\right) (f(a) + f(b)) \right| \leq \frac{\gamma}{(\nu+1)} \frac{(b-a)^{\nu+1}}{2^\nu}. \quad (33)$$

The theorem is proved in all cases. $\square$

We give

Theorem 5. Let $\nu \geq 1$, $n = \lceil \nu \rceil$, and $f \in AC^n([a, b])$. We assume that $D_{*a}^\nu f, D_{b-}^\nu f \in L_1([a, b])$. Then

i)

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} [(t-a)^{\nu} + (b-t)^{\nu}], \quad (34)$$

$\forall t \in [a, b],$

ii) when $\nu = 1$, from (34), we have

$$\left| \int_a^b f(x) dx - [f(a)(t-a) + f(b)(b-t)] \right| \leq \|f'\|_{L_1([a,b])} (b-a), \quad \forall t \in [a, b], \quad (35)$$

iii) from (35), we obtain ($\nu = 1$ case)

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \|f'\|_{L_1([a,b])} (b-a), \quad (36)$$

iv) at $t = \frac{a+b}{2}$, $\nu > 1$, the right hand side of (34) is minimized, and we get:

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}, \quad (37)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, \dots, n-1$; $\nu > 1$, from (37), we obtain

$$\left| \int_a^b f(x) dx \right| \leq \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}, \quad (38)$$

which is a sharp inequality,

vi) more generally, for $j = 0, 1, 2, \dots, N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left(\frac{b-a}{N} \right)^{\nu} [j^{\nu} + (N-j)^{\nu}], \quad (39)$$

vii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \dots, n-1$, from (39) we get:

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left(\frac{b-a}{N} \right)^{\nu} [j^{\nu} + (N-j)^{\nu}], \quad (40)$$

$j = 0, 1, 2, \dots, N$,

viii) when $N = 2$ and $j = 1$, (40) turns to

$$\left| \int_a^b f(x) dx - \frac{(b-a)}{2} (f(a) + f(b)) \right| \leq \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}. \quad (41)$$

Proof. Here $\nu \geq 1$ and $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_1([a, b])$. By (10) we get

$$\begin{aligned} \left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| &\leq \frac{1}{\Gamma(\nu)} (x-a)^{\nu-1} \int_a^x |D_{*a}^{\nu} f(t)| dt \\ &\leq \frac{(x-a)^{\nu-1}}{\Gamma(\nu)} \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \end{aligned} \quad (42)$$

$\forall x \in [a, b]$.

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\|D_{*a}^{\nu} f\|_{L_1([a,b])}}{\Gamma(\nu)} (x-a)^{\nu-1}, \quad (43)$$

$\forall x \in [a, b]$.

By (11) we get

$$\begin{aligned} \left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| &\leq \frac{1}{\Gamma(\nu)} (b-x)^{\nu-1} \int_x^b |D_{b-}^{\nu} f(z)| dz \\ &\leq \frac{(b-x)^{\nu-1}}{\Gamma(\nu)} \|D_{b-}^{\nu} f\|_{L_1([a,b])}, \end{aligned} \quad (44)$$

$\forall x \in [a, b]$.

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\|D_{b-}^{\nu} f\|_{L_1([a,b])}}{\Gamma(\nu)} (b-x)^{\nu-1}, \quad (45)$$

$\forall x \in [a, b]$.

Call

$$\delta_1 := \frac{\|D_{*a}^{\nu} f\|_{L_1([a,b])}}{\Gamma(\nu)}, \quad (46)$$

and

$$\delta_2 := \frac{\|D_{b-}^{\nu} f\|_{L_1([a,b])}}{\Gamma(\nu)}. \quad (47)$$

Set

$$\delta := \max(\delta_1, \delta_2). \quad (48)$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \delta (x-a)^{\nu-1}, \quad (49)$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \delta (b-x)^{\nu-1}, \quad (50)$$

$\forall x \in [a, b]$.

As in the proof of Theorem 4, we get:

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq \frac{\delta}{\nu} [(t-a)^{\nu} + (b-t)^{\nu}], \quad (51)$$

$\forall t \in [a, b]$.

The rest of the proof is similar to the proof of Theorem 4. $\square$

We continue with

Theorem 6. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$; $f \in AC^n([a, b])$, with $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_q([a, b])$. Then

i)

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_q([a,b])}, \|D_{b-}^{\nu} f\|_{L_q([a,b])} \right\}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1) + 1)^{\frac{1}{p}}} \left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}} \right], \quad (52)$$

$\forall t \in [a, b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (52) is minimized, and we get:

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_q([a,b])}, \|D_{b-}^{\nu} f\|_{L_q([a,b])} \right\} (b-a)^{\nu+\frac{1}{p}}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}} \frac{1}{2^{\nu-\frac{1}{q}}}, \quad (53)$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, \dots, n-1$, we obtain

$$\left| \int_a^b f(x) dx \right| \leq \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_q([a,b])}, \|D_{b-}^{\nu} f\|_{L_q([a,b])} \right\} (b-a)^{\nu+\frac{1}{p}}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}} \frac{1}{2^{\nu-\frac{1}{q}}}, \quad (54)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, \dots, N \in \mathbb{N}$, it holds

$$\begin{aligned} & \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \\ & \leq \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_q([a,b])}, \|D_{b-}^{\nu} f\|_{L_q([a,b])} \right\} (b-a)^{\nu+\frac{1}{p}}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \end{aligned} \quad (55)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \dots, n-1$, from (55) we get:

$$\begin{aligned} & \left| \int_a^b f(x) dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq \\ & \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_q([a,b])}, \|D_{b-}^{\nu} f\|_{L_q([a,b])} \right\} (b-a)^{\nu+\frac{1}{p}}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \end{aligned} \quad (56)$$

for $j = 0, 1, 2, \dots, N$,

vi) when $N = 2$ and $j = 1$, (56) turns to

$$\begin{aligned} & \left| \int_a^b f(x) dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \\ & \frac{\max \left\{ \|D_{*a}^{\nu} f\|_{L_q([a,b])}, \|D_{b-}^{\nu} f\|_{L_q([a,b])} \right\} (b-a)^{\nu+\frac{1}{p}}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}} \frac{1}{2^{\nu-\frac{1}{q}}}, \end{aligned} \quad (57)$$

vii) when $1/q < \nu \leq 1$, inequality (57) is again valid but without any boundary conditions.

Proof. Here $\nu > 0$, $n = \lceil \nu \rceil$, $f \in AC^n([a, b])$; $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, with $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_q([a, b])$. By (10) we have

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} |D_{*a}^{\nu} f(t)| dt \leq$$

$$\begin{aligned} \frac{1}{\Gamma(\nu)} \left(\int_a^x (x-t)^{p(\nu-1)} dt \right)^{\frac{1}{p}} \left(\int_a^x |D_{*a}^\nu f(t)|^q dt \right)^{\frac{1}{q}} \leq \\ \frac{1}{\Gamma(\nu)} \frac{(x-a)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \|D_{*a}^\nu f\|_{L_q([a,b])}. \end{aligned} \quad (58)$$

Here we assume that $\nu > \frac{1}{q} \Leftrightarrow p(\nu-1)+1 > 0$. So, we get

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\|D_{*a}^\nu f\|_{L_q([a,b])}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}}} (x-a)^{\nu-\frac{1}{q}}, \quad (59)$$

$\forall x \in [a, b]$.

By (11) we have

$$\begin{aligned} \left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} |D_{b-}^\nu f(z)| dz \leq \\ \frac{1}{\Gamma(\nu)} \left(\int_x^b (z-x)^{p(\nu-1)} dz \right)^{\frac{1}{p}} \left(\int_x^b |D_{b-}^\nu f(z)|^q dz \right)^{\frac{1}{q}} \leq \\ \frac{1}{\Gamma(\nu)} \frac{(b-x)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \|D_{b-}^\nu f\|_{L_q([a,b])}. \end{aligned} \quad (60)$$

So, we get

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\|D_{b-}^\nu f\|_{L_q([a,b])}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}}} (b-x)^{\nu-\frac{1}{q}}, \quad (61)$$

$\forall x \in [a, b]$.

Call

$$\rho_1 := \frac{\|D_{*a}^\nu f\|_{L_q([a,b])}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}}}, \quad (62)$$

and

$$\rho_2 := \frac{\|D_{b-}^\nu f\|_{L_q([a,b])}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}}}. \quad (63)$$

Set

$$\rho := \max(\rho_1, \rho_2), \quad m := \nu - \frac{1}{q} > 0. \quad (64)$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \rho (x-a)^m, \quad (65)$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \rho (b-x)^m, \quad (66)$$

$\forall x \in [a, b]$.

As in the proof of Theorem 4, we obtain:

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq$$

$$\frac{\rho}{(m+1)!} \left[(t-a)^{m+1} + (b-t)^{m+1} \right] =$$

$$\frac{\max \left\{ \|D_{*a}^\nu f\|_{L_q([a,b])}, \|D_{b-}^\nu f\|_{L_q([a,b])} \right\}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p} \right)} \left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}} \right], \quad (67)$$

$\forall t \in [a, b]$.

The rest of the proof is similar to the proof of Theorem 4. $\square$

References

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